A Parker-Forney-Traub-type algorithm for Q/S-Hessenberg-Vandermonde matrices

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Outline

A Parker-Forney-Traub-type algorithm for Q/S-Hessenberg-Vandermonde matrices

- Quasiseparable matrices & Special cases
- A Class of Polynomials
- Recurrence relations & Complexity

A Björck-Pereyra-type algorithm for Szegö-Vandermonde matrices

- Classical Björck-Pereyra algorithm
- Algorithm for Szegö-Vandermonde matrices
- Numerical experiments

Equivalence of Hadamard & PN Matrices

- Hadamard-Sylvester matrices
- Pseudo-Noise matrices
- Equivalence

A Parker-Forney-Traub-type algorithm for Quasiseparable-Hessenberg-Vandermonde matrices

Joint work with V.Olshevsky

Quasiseparable Matrices

- **Definition.** A matrix C is **quasiseparable of order one** if

$$\max \text{Rank} C_{12} = \max \text{Rank} C_{21} = 1$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline C_{21} & * \end{array} \right]$$

- **Previous Work.** Gohberg-Kaashoek-Lerer, Dewilde, Gohberg-Eidelman, Van Barel et al, Tyrtyshnikov et al, Bini et al.

- Associate with a matrix C a system of polynomials R consisting of the **characteristic polynomials** of principal submatrices of C:

$$r_k(x) = \det(R_{k \times k} - xI)$$

Generators of a Quasiseparable matrix

- Can be represented in terms of their **generators**:

Diagonal entries

$$d_k \quad k = 1, \dots, n$$

Lower Generators		Upper Generators	
p_k	$k = 2, \dots, n$	g_k	$k = 1, \dots, n-1$
a_k	$k = 2, \dots, n-1$	b_k	$k = 2, \dots, n-1$
q_k	$k = 1, \dots, n-1$	h_k	$k = 2, \dots, n$

- **Example.** In terms of generators, with $n = 5$,

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

Important Special Cases

Tridiagonal

- For $a_k = b_k = 0$, $p_k = h_k = 1$, the matrix becomes **tridiagonal**:

$$\begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

- **Corresponding polynomial system:** Polynomials orthogonal on a real interval.

Important Special Cases

Unitary Hessenberg

➠ For $a_k = 0$, then a quasiseparable matrix of the following form is obtained:

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & p_4q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & p_5q_4 & d_5 \end{bmatrix}$$

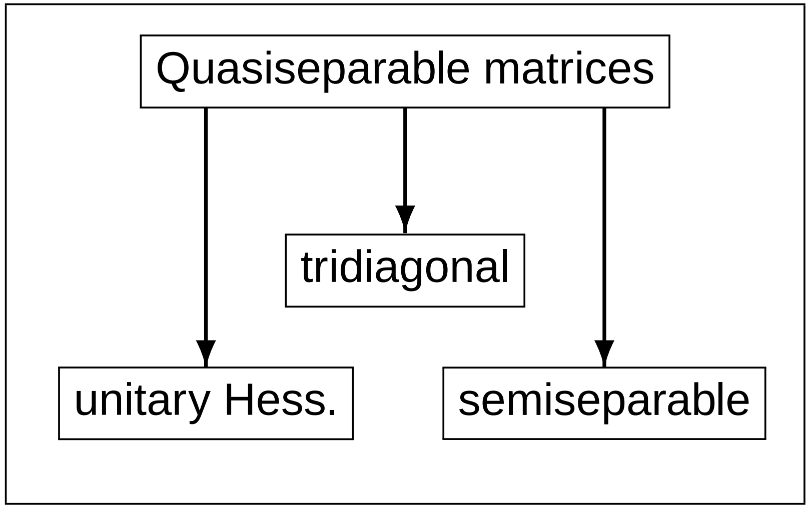
➠ **Corresponding polynomial system:** Polynomials orthogonal on the unit circle (**Szegö**).

A Class of Polynomials

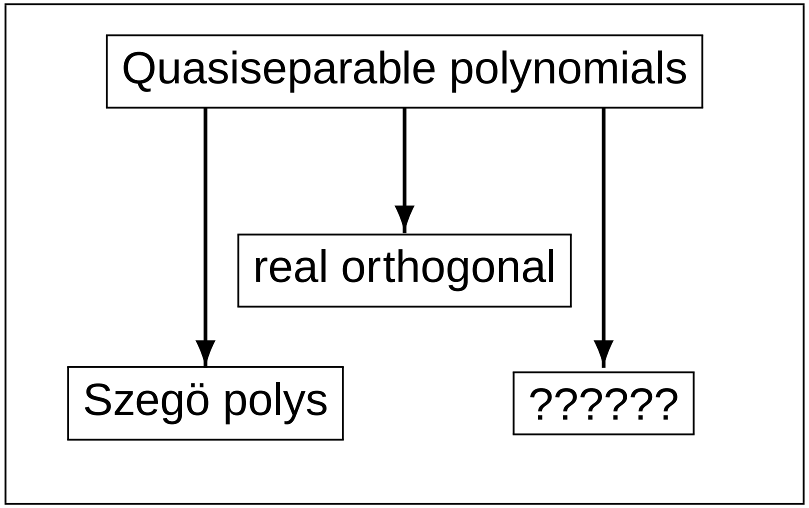
- Systems of polynomials associated with a **quasiseparable matrix**.

(**Quasiseparable polynomials**)?

Tridiagonal matrices, unitary Hessenberg matrices, semiseparable matrices are special cases of quasiseparable matrices	$\Rightarrow$	Polynomials orthogonal on (i) a real interval, and (ii) on the unit circle are special cases of quasiseparable polynomials



$\Rightarrow$



Fast Parker-Forney-Traub-like algorithms

- Taking advantage of structure, **fast algorithms** are possible:

Parker-Forney-Traub (1964,1966)	Vandermonde matrices	$O(n^2)$
Gohberg-Olshevsky (1994)	Chebyshev-Vandermonde matrices	$O(n^2)$
Calvetti-Reichel (1993)	three-term Vandermonde matrices	$O(n^2)$
Gohberg-Olshevsky (1996)	Vandermonde-like matrices	$O(n^2)$
Olshevsky (2001)	Szegö-Vandermonde matrices	$O(n^2)$
	Q/S-Hessenberg-Vandermonde matrices	???

- Gaussian elimination ignores these structures, and requires $O(n^3)$ operations.

Parker-Forney-Traub-like Algorithm

- **Parker-Forney-Traub-like algorithm**. Based on the formula

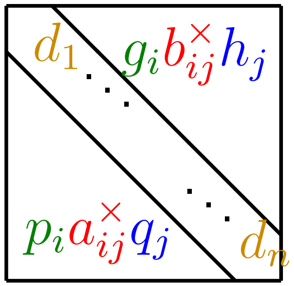
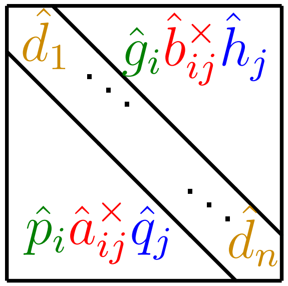
$$V_P{}^{-1} = \begin{bmatrix} p_0(x_1) & p_1(x_1) & \cdots & p_{n-1}(x_1) \\ p_0(x_2) & p_1(x_2) & \cdots & p_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_0(x_n) & p_1(x_n) & \cdots & p_{n-1}(x_n) \end{bmatrix}^{-1} = \tilde{I} \cdot V_{\hat{P}}{}^{T} \cdot \text{diag}(c_1, c_2, \ldots, c_n)$$

where $\tilde{I}$ is the antidiagonal matrix, $c_k = \prod_{\substack{j=1 \\ j \neq k}}^{n} (x_j - x_k)^{-1}$

- $\hat{P}$ is the system of **Horner-like** polynomials corresponding to the polynomial system P. (When P is the monomial basis, this is the classical Parker-Forney-Traub (1964,1966))
- **What is the computational cost of this algorithm?**

Horner-like Polynomials

- **Theorem.** Let C be a Hessenberg, quasiseparable of order one matrix with associated system of polynomials R. Then the system of **Horner-like** polynomials $\hat{R}$ corresponds to the matrix $\tilde{I} \cdot C^T \cdot \tilde{I}$.

		
Q/S-Hessenberg Matrix	$\Rightarrow$	Q/S-Hessenberg Matrix
$\{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$		$\{\hat{r}_0(x), \hat{r}_1(x), \ldots, \hat{r}_{n-1}(x)\}$
Polynomial System		Horner-like polynomials

Recurrence Relations

➠ **Theorem.** Let P be a system of quasiseparable polynomials with $\deg p_k(x) = k$ and generators $d_k, p_k, a_k, q_k, g_k, b_k, h_k$. Let $\{x_k\}$ be a set of n distinct nodes, and β_k the coefficients of the polynomial $\beta(x) = \prod_{k=1}^{n}(x - x_k)$ decomposed in the P basis. Then the Horner-like polynomials $\hat{P}$ satisfy the four-term recurrence relations

$$\hat{p}_k(x) = \psi_k(x)\hat{p}_{k-1}(x) - \varphi_k(x)\hat{p}_{k-2}(x) - z_k$$

$$\psi_k(x) = (\hat{d}_k - x) - \frac{\hat{p}_k \hat{h}_k \hat{q}_{k-1} \hat{b}_{k-1}}{\hat{h}_{k-1}} \qquad \varphi_k(x) = \frac{\hat{p}_k \hat{h}_k \hat{q}_{k-1}}{\hat{h}_{k-1}} \left((\hat{d}_k - x)\hat{b}_k - \hat{h}_k \hat{g}_k \right)$$

$$z_k = \frac{\hat{h}_k \hat{b}_{k-1}}{\hat{p}_{k+1} \hat{q}_k \hat{h}_{k-1}} \beta_{n-k+1} - \frac{1}{\hat{p}_{k+1} \hat{q}_k} \beta_{n-k}$$

with $\hat{d}_k = d_{n-k+1}, \hat{p}_k = q_{n-k+1}, \hat{a}_k = a_{n-k+1}, \hat{q}_k = p_{n-k+1}, \hat{g}_k = h_{n-k+1}, \hat{b}_k = b_{n-k+1}, \hat{h}_k = g_{n-k+1}$.

Complexity

- The coefficients of $\beta(x)$ in the basis P can be computed in $O(n)$ operations.
- Each Horner-like polynomial can be evaluated at a given node in $O(n)$ operations.
- The total cost of the algorithm is $O(n^2)$ operations. Comparing this to the complexity of Gaussian elimination, $O(n^3)$, we have the algorithm is **FAST!**

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Joint work with V.Olshevsky

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Joint work with Y.Eidelman, I.Gohberg, I.Koltracht, and V.Olshevsky

The Björck-Pereyra Algorithm

- The **Björck-Pereyra algorithm** (1970) is based on the formula

$$V^{-1} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}^{-1} = U_1^{-1} \ldots U_{n-1}^{-1} L_{n-1}^{-1} \ldots L_1^{-1},$$

with

$$U_k^{-1} = \begin{bmatrix} 1 & -\gamma_1^{(k)} & & \\ & 1 & \ddots & \\ & & \ddots & -\gamma_n^{(k)} \\ & & & 1 \end{bmatrix}, \qquad L_k^{-1} = \begin{bmatrix} \delta_0^{(k)} & & & \\ & \delta_1^{(k)} & & \\ & & \ddots & \\ & & & \delta_n^{(k)} \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix},$$

- **Fast:** requires only $O(n^2)$ arithmetic operations vs $O(n^3)$ of Gaussian elimination.

Björck-Pereyra-like Algorithms

Tang-Golub (1981)	block Vandermonde matrices
Reichel-Opfer (1991)	Chebyshev-Vandermonde matrices
Higham (1988,90)	three-term Vandermonde matrices

- If the polynomials are Szëgo polynomials, the resulting matrix is a Szëgo-Vandermonde matrix.

$$V_\Phi = \begin{bmatrix} \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_{n-1}(x_1) \\ \phi_0(x_2) & \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_{n-1}(x_2) \\ \phi_0(x_3) & \phi_1(x_3) & \phi_2(x_3) & \cdots & \phi_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \phi_2(x_n) & \cdots & \phi_{n-1}(x_n) \end{bmatrix}$$

Formula

- based on the formula

$$V_\Phi^{-1} = \begin{bmatrix} \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_{n-1}(x_1) \\ \phi_0(x_2) & \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_{n-1}(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \phi_2(x_n) & \cdots & \phi_{n-1}(x_n) \end{bmatrix}^{-1} = U_1^{-1} \ldots U_{n-1}^{-1} L_{n-1}^{-1} \ldots L_1^{-1},$$

with

$$U_k = \left[\begin{array}{c|ccc} \frac{1}{\alpha_0} & & & \\ 0 & & & \\ \vdots & & \mathbf{H - x_k I} & \\ 0 & & & \\ \hline 0 & 0 & \cdots \; 0 & \frac{1}{\alpha_{n-k}} \end{array}\right], \qquad L_k^{-1} = \begin{bmatrix} \delta_0^{(k)} & & & \\ & \delta_1^{(k)} & & \\ & & \ddots & \\ & & & \delta_n^{(k)} \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix},$$

- H is **Unitary Hessenberg**.

Unitary Hessenberg Matrices

$$H = \begin{bmatrix} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & \cdots & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & \cdots & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ 0 & \mu_2 & \cdots & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{n-1}\rho_{n-2}^* & \vdots \\ 0 & \cdots & \cdots & 0 & \mu n - 1 & -\rho_n\rho_{n-1}^* \end{bmatrix}$$

- For monomials (classical BP), the algorithm is **fast** because the H's are **bidiagonal**.
- For the Szegö case, the algorithm is **fast** using **factorizations** of H.
  1. Factorization of H into **Givens rotations**
  2. Factorization of $H - x_k I$ using **quasiseparable structure**

Decomposition of H into Givens (plane) Rotations

- The details of the decomposition of H into **Givens (plane) rotations**:

$$H = H(\rho_1) \times H(\rho_2) \times \cdots \times H(\rho_{n-1}) \times \tilde{H}(\rho_n)$$

where

$$H(\rho_k) = \text{diag}\left\{ I_{k-1}, \begin{bmatrix} \rho_k & \mu_k \\ \mu_k & -\rho_k^* \end{bmatrix}, I_{n-k-1} \right\}$$

$$\tilde{H}(\rho_k) = \text{diag}\left\{ I_{n-1}, \rho_k \right\}$$

Decomposition of H into Implicit Shifts

- The details of a factorization of H using **Implicit Shifts**:

$$H - x_k I = R_0 \times R_1 \times \cdots \times R_{n-1} \times R_n$$

where

$$R_0 = \text{diag}\left\{\begin{bmatrix} 1 & -\rho_0^* \end{bmatrix}, I_{n-1}\right\} \qquad R_n = \text{diag}\left\{I_{n-1}, \begin{bmatrix} -x_n \\ \rho_n \end{bmatrix}\right\}$$

$$R_k = \text{diag}\left\{I_{k-1}, \begin{bmatrix} -x_k & 0 & 0 \\ \rho_k & 0 & \mu_k \\ \mu_k & 1 & -\rho_k^* \end{bmatrix}, I_{n-k-1}\right\} \quad k = 1, 2, \ldots, n-1$$

- This factorization uses the **quasiseparable structure** of $H - x_k I$.

Numerical Illustrations - 30×30 Matrices

- We compare the **forward accuracy** of $\hat{x}$ from MATLAB in double precision by

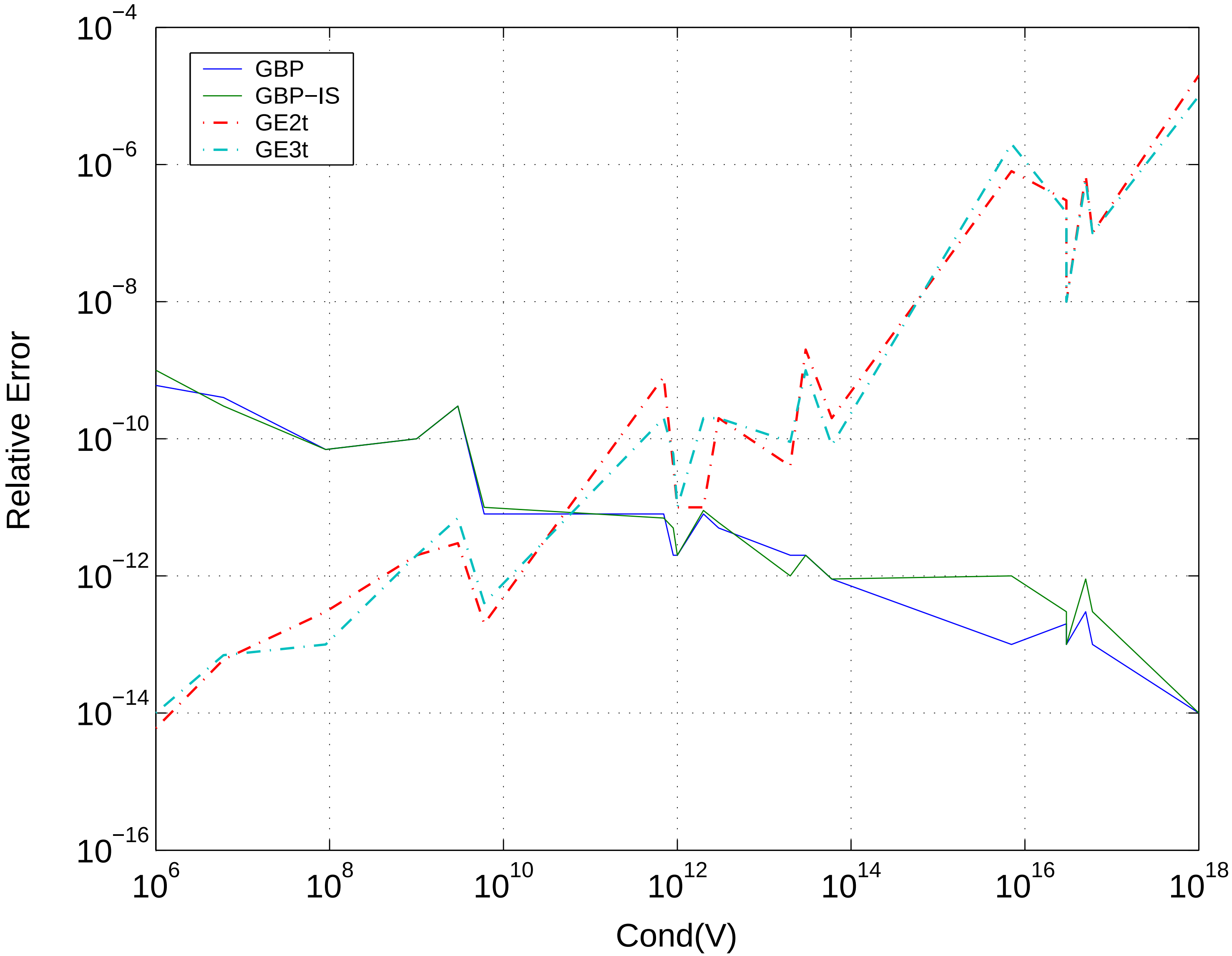
$$e = \frac{\|x - \hat{x}\|_2}{\|x\|_2},$$

x is "exact" solution from Maple v7 using software-implemented 40-digit arithmetic.

- $\log e$ gives estimate of number of correct significant figures out of 16.
- The algorithms
  - **GBP**/**GBP-IS** - Generalized BP alg. using plane rotations and implicit shifts, resp.
  - **GE2t**/**GE3t** - Gaussian elimination with matrix derived from 2-term and 3-term recurrence relations, resp.
- The nodes $\{x_k\}_{k=1}^{30}$ are ordered using **Leja ordering**.
(See L. Reichel et al)

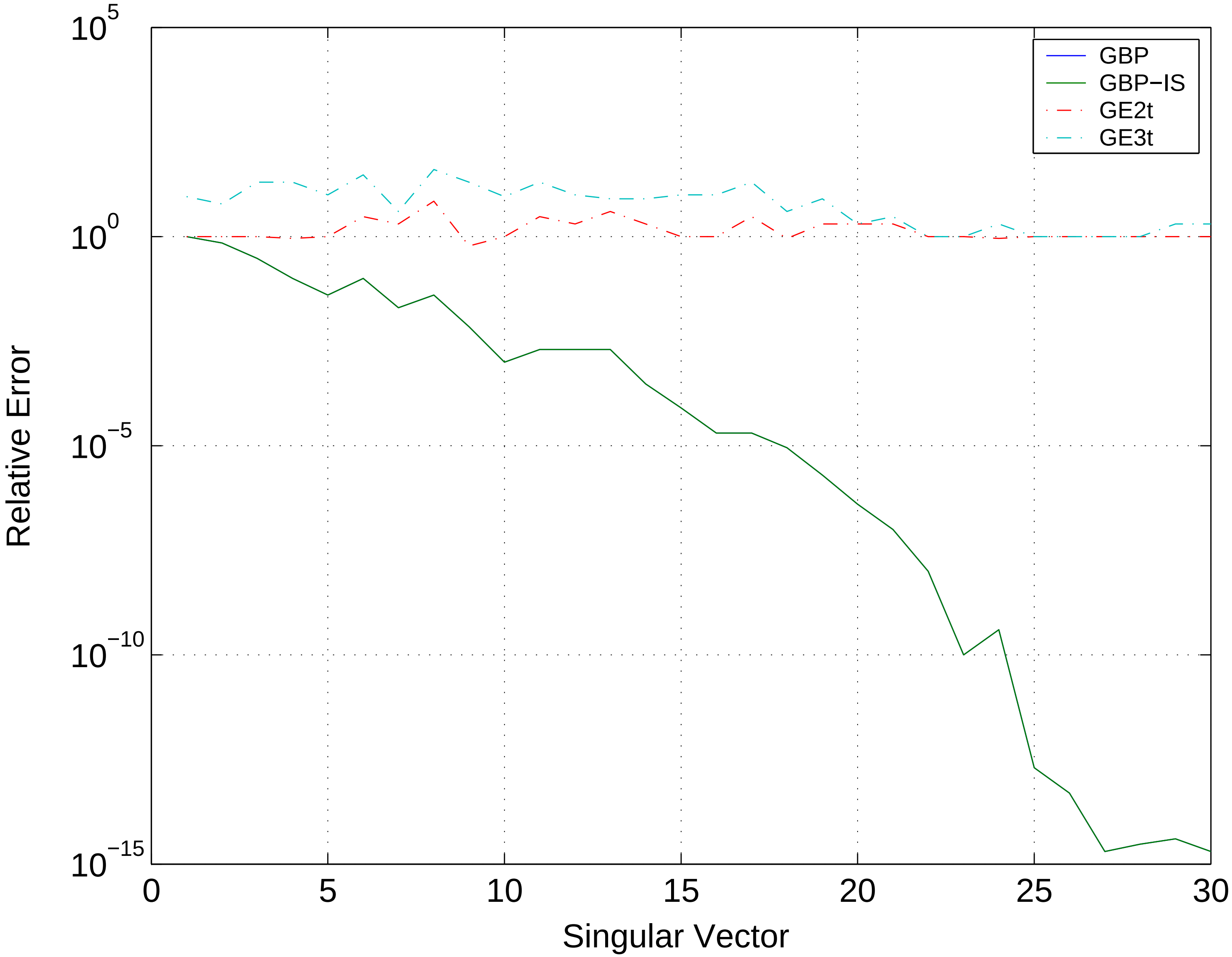
Numerical Illustrations - Experiment 1

Effects of Conditioning



Numerical Illustrations - Experiment 2

Dependence on Direction



Numerical Illustrations - Experiment 3

Iterative Refinement

		no refinement		iterative refinement			
#	cond(V)	GBP	GBP-IS	GBP	GBP-IS	GE2t	GE3t
1	1e+07	2e-09	4e-09	1e-14	9e-15	4e-13	6e-13
2	3e+07	2e-10	2e-10	4e-14	7e-15	4e-14	8e-14
3	1e+08	2e-10	2e-10	4e-14	2e-14	9e-13	2e-13
4	6e+08	6e-11	6e-11	3e-14	1e-14	4e-13	4e-13
5	1e+07	3e-10	5e-10	4e-15	4e-15	1e-13	2e-13
6	5e+05	5e-10	3e-10	1e-15	9e-16	4e-14	7e-14
7	3e+06	3e-10	1e-10	1e-15	1e-15	2e-14	2e-13
8	1e+08	7e-10	6e-10	3e-15	4e-15	4e-14	3e-14
9	2e+07	8e-10	5e-10	2e-15	3e-15	1e-13	9e-14
10	3e+07	5e-10	5e-10	3e-14	3e-14	8e-14	1e-13

Numerical Illustrations - Summary

- **Experiment 1** - As the condition number is raised, **GE** gives higher errors, but the **GBP** & **GBP-IS** give lower errors.
- **Experiment 2** - Unlike **GE** which is insensitive to the direction of the RHS vector, **GBP** & **GBP-IS** are affected by the direction of the RHS vector.
- **Experiment 3** - With one step of **iterative refinement**, the **GBP** & **GBP-IS** both perform very well in the previously noted cases where performance was poor (low condition number).

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Joint work with Y.Eidelman, I.Gohberg, I.Koltracht, and V.Olshevsky

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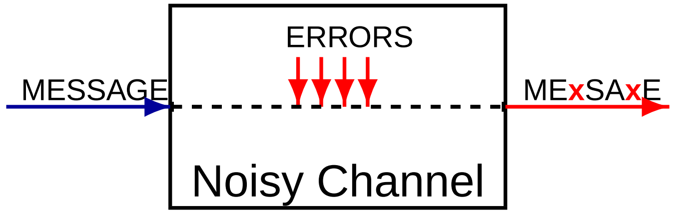
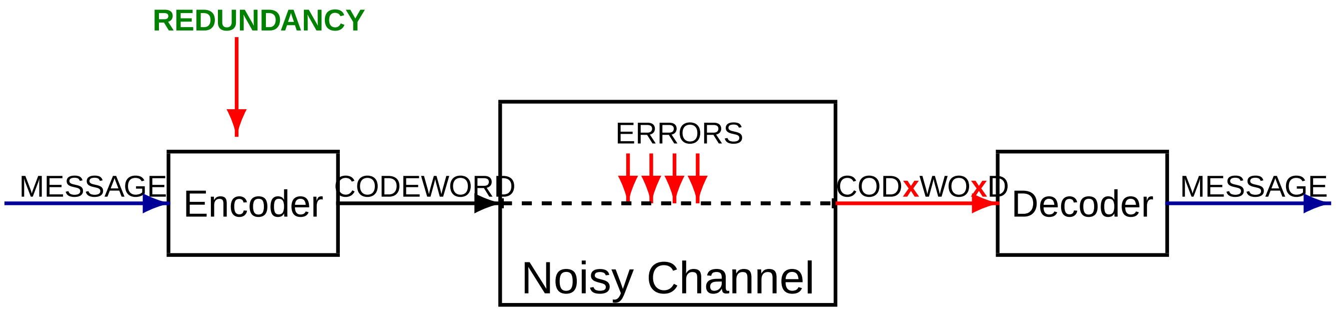
Equivalence of Hadamard and Pseudo-Noise Matrices

Joint work with V.Olshevsky and L.Sakhnovich

Transmission over Noisy Channel, No Coding Theory



Transmission over Noisy Channel, With Coding Theory



Hadamard Matrices

Hadamard matrices of size n x n, are $(-1, 1)$ matrices such that

$$H_n^T H_n = nI_n$$

A special case: **Hadamard-Sylvester matrices**

$$H_1 = [1], \qquad H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

For example,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Linear Recurring Sequences. A Matrix Interpretation

- Linear **m-term** recurrence relation:

$$\mathbf{a_i} = \mathbf{a_{i-1}}\mathbf{h_{m-1}} + \mathbf{a_{i-2}}\mathbf{h_{m-2}} + \cdots + \mathbf{a_{i-m+1}}\mathbf{h_1} + \mathbf{a_{i-m}}\mathbf{h_0} \quad \text{for } i \geq m$$

- $m \times m$ Matrix formulation

$$\begin{bmatrix} \mathbf{a_{i-(m-1)}} \\ \mathbf{a_{i-(m-2)}} \\ \mathbf{a_{i-(m-3)}} \\ \vdots \\ \mathbf{a_{i-1}} \\ \mathbf{a_i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ \mathbf{h_0} & \mathbf{h_1} & \mathbf{h_2} & \ldots & \mathbf{h_{m-2}} & \mathbf{h_{m-1}} \end{bmatrix} \begin{bmatrix} \mathbf{a_{i-(m)}} \\ \mathbf{a_{i-(m-1)}} \\ \mathbf{a_{i-(m-2)}} \\ \vdots \\ \mathbf{a_{i-2}} \\ \mathbf{a_{i-1}} \end{bmatrix}$$

- The characteristic polynomial of **degree m**:

$$h(x) = x^m + \mathbf{h_{m-1}}x^{m-1} + \cdots + \mathbf{h_1}x + \mathbf{h_0}$$

PN Sequences

- The sequence $\mathbf{a_0 a_1 a_2 a_3} \ldots$ of m-term recurrence relations is **periodic**:

$$\text{period } \leq \mathbf{2^m - 1}.$$

- **Definition.** A sequence with **equality**

$$\text{period } = \mathbf{2^m - 1}.$$

is called a **Pseudo-Noise sequence**.

- For $h(x) = x^4 + x^3 + 1$ (i.e., $m = 4$), and the initial state $a_0 a_1 a_2 a_3 = 1000$, the resulting PN Sequence is given by

$$\underbrace{100011110101100}_{\text{period } 15}\underbrace{100011110101100}_{\text{period } 15}\underbrace{100011110101100}_{\text{period } 15}\ldots\ldots$$

PN Matrices

- For the **PN Sequence** listed above

100011110101100 100011110101100 100011110101100 ...

$$
T = \left[\begin{array}{c|cccc|ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
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0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}\right]
$$

The Equivalence

- **Theorem.** The $(0,1)$ Hadamard-Sylvester matrices and the $(0,1)$ PN matrices are equivalent, i.e., they can be obtained one from another via row and column permutations.
- **Sakhnovich(1998)** proved this result for $n = 16$ using combinatorial tricks.

Rank Structure of the Hadamard Matrices

- **Theorem.** Let H_n be an arbitrary $n \times n$ Hadamard matrix

$$H_n \cdot H_n^T = nI.$$

Then there are two cases:

- If n is divisible by 8 then

$$\text{Rank}\widetilde{H_n} \leq \frac{n}{2}$$

- If n not divisible by 8 then

$$\text{Rank}\widetilde{H_n} = n - 1.$$

where $\widetilde{H_n}$ denotes the $(0,1)$ Hadamard matrix.

Equivalence of Hadamard and Pseudo-Noise Matrices

Joint work with V.Olshevsky and L.Sakhnovich

Supplemental Slides

Confederate Matrices

- **Definition** For polynomials $R = \{r_0(x), r_1(x), \ldots, r_n(x)\}$ satisfying n-term recurrence relations

$$r_k(x) = \alpha_k \cdot x r_{k-1}(x) - a_{k-1,k} \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \cdots - a_{0,k} \cdot r_0(x).$$

and the polynomial

$$b(x) = b_0 \cdot r_0(x) + b_1 \cdot r_1(x) + \cdots + b_{n-1} \cdot r_{n-1}(x) + b_n \cdot r_n(x)$$

define the **confederate matrix** of b with respect to R by

$$C_R(b) = \begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n} \end{bmatrix}$$

Confederate Matrices

$$
\begin{aligned}
r_1(x) &= \alpha_1 \cdot x r_0(x) - a_{0,1} \cdot r_0(x) \\
r_2(x) &= \alpha_2 \cdot x r_1(x) - a_{1,2} \cdot r_1(x) - a_{0,2} \cdot r_0(x) \\
r_3(x) &= \alpha_3 \cdot x r_2(x) - a_{2,3} \cdot r_2(x) - a_{1,3} \cdot r_1(x) - a_{0,3} \cdot r_0(x) \\
&\vdots
\end{aligned}
$$

$$
\begin{bmatrix}
\frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\
\frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\
0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\
0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n}
\end{bmatrix}
$$

Confederate Matrices

$$\begin{aligned}
r_1(x) &= \alpha_1 \cdot x r_0(x) - a_{0,1} \cdot r_0(x) \\
r_2(x) &= \alpha_2 \cdot x r_1(x) - a_{1,2} \cdot r_1(x) - a_{0,2} \cdot r_0(x) \\
r_3(x) &= \alpha_3 \cdot x r_2(x) - a_{2,3} \cdot r_2(x) - a_{1,3} \cdot r_1(x) - a_{0,3} \cdot r_0(x) \\
&\vdots
\end{aligned}$$

$$\begin{bmatrix}
\frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\
\frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\
0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\
0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n}
\end{bmatrix}$$

Confederate Matrices

$$
\begin{aligned}
r_1(x) &= \alpha_1 \cdot x r_0(x) - a_{0,1} \cdot r_0(x) \\
r_2(x) &= \alpha_2 \cdot x r_1(x) - a_{1,2} \cdot r_1(x) - a_{0,2} \cdot r_0(x) \\
r_3(x) &= \alpha_3 \cdot x r_2(x) - a_{2,3} \cdot r_2(x) - a_{1,3} \cdot r_1(x) - a_{0,3} \cdot r_0(x) \\
&\vdots
\end{aligned}
$$

$$
\begin{bmatrix}
\frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\
\frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\
0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\
0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n}
\end{bmatrix}
$$

Motivation for Horner-like Polynomials

- The confederate matrix $C(b)$ for a polynomial $b(x) = b_0 + b_1 x + \cdots + b_n x_n$ in the **monomial basis** reduces to the companion matrix, and the confederate matrix $C_R(\hat{p}_n)$ for the **Horner polynomials** is:

$$\mathbf{C(b)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -b_{n-1} \end{bmatrix} \quad \mathbf{C_R(\hat{p}_n)} = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -b_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

- **Observation**: $\mathbf{C_R(\hat{p}_n)} = \tilde{I} \cdot \mathbf{C(b)}^T \cdot \tilde{I}$

Unitary Hessenberg Matrices

$$H = \begin{bmatrix} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & \dots & \dots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & \dots & \dots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ 0 & \mu_2 & \dots & \dots & -\rho_{n-1}\mu_{n-2}\cdots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{n-1}\rho_{n-2}^* & \vdots \\ 0 & \dots & \dots & 0 & \mu n-1 & -\rho_n\rho_{n-1}^* \end{bmatrix}$$

Definition of $^{\#}$ Notation

- Following notation commonly used in signal processing literature, we denote

$$f_k^{\#}(z) = z^k \left[f_k\left(\frac{1}{z^*}\right)\right]^*$$

- This **reverses the order** of the coefficients, and takes **complex conjugates**.
- **Example:** For $n = 3$,

$$f(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

$$f(z)^{\#} = a_0^* z^3 + a_1^* z^2 + a_2^* z + a_3^*$$

The Szegö Polynomials

- **Orthogonal on the unit circle** with respect to some weight function.

$$< p(x), q(x) >= \frac{1}{2\pi}\int_{-\pi}^{\pi} p(e^{i\theta}) \cdot [q(e^{i\theta})]^* w^2(\theta) d\theta.$$

- Satisfy **two-term recurrence relations**

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_{i+1}(x) \\ \phi_{i+1}^{\#}(x) \end{bmatrix} = \frac{1}{\mu_{i+1}}\begin{bmatrix} 1 & -\rho_{i+1} \\ -\rho_{i+1}^* & 1 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \phi_i(x) \\ \phi_i^{\#} \end{bmatrix},$$

where ρ_k are the **reflection coefficients**, and

$$\mu_k = \begin{cases} 1 & |\rho_k| = 1 \\ \sqrt{1 - |\rho_k|^2} & \text{otherwise} \end{cases}$$

are the **complementary parameters**

Three-Term Recurrence Relations for Szegö Polynomials

$$\phi_0(x) = 1, \qquad \phi_1(x) = \frac{1}{\mu_1}(x \cdot \phi_0(x) - \rho_1 \cdot \phi_0(x)),$$

$$\phi_k(x) = [\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}] \cdot \phi_{k-1}(x) - \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \phi_{k-2}(x).$$

Factorizations of the Unitary Hessenberg Matrix H

- Decomposition of H into the product of Givens (plane) rotations.

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} = \begin{bmatrix} * & * & & & \\ * & * & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & * & * & & \\ & * & * & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \cdots$$

- Factorization of $H - x_k I$ capturing the shifts.

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & & \\ * & 0 & * & & \\ * & 1 & * & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & * & 0 & 0 & \\ & * & 0 & * & \\ & * & 1 & * & \\ & & & & 1 \end{bmatrix} \cdots$$

Decomposition of H into Plane Rotations

- The details of the decomposition of H into plane rotations:

$$H = H(\rho_1) \times H(\rho_2) \times \cdots \times H(\rho_{n-1}) \times \tilde{H}(\rho_n)$$

where

$$H(\rho_k) = \text{diag}\left\{ I_{k-1}, \begin{bmatrix} \rho_k & \mu_k \\ \mu_k & -\rho_k^* \end{bmatrix}, I_{n-k-1} \right\}$$

$$\tilde{H}(\rho_k) = \text{diag}\left\{ I_{n-1}, \rho_k \right\}$$

Decomposition of H into Implicit Shifts

- The details of a factorization of H using **Implicit Shifts**:

$$H - x_k I = R_0 \times R_1 \times \cdots \times R_{n-1} \times R_n$$

where

$$R_0 = \text{diag}\left\{ \begin{bmatrix} 1 & -\rho_0^* \end{bmatrix}, I_{n-1} \right\}$$

$$R_k = \text{diag}\left\{ I_{k-1}, \begin{bmatrix} -x_k & 0 & 0 \\ \rho_k & 0 & \mu_k \\ \mu_k & 1 & -\rho_k^* \end{bmatrix}, I_{n-k-1} \right\}$$

for $k = 1, 2, \ldots, n-1$, and

$$R_n = \text{diag}\left\{ I_{n-1}, \begin{bmatrix} -x_n \\ \rho_n \end{bmatrix} \right\}$$

Classical Vandermonde Systems

- **Vandermonde** - Linear systems $Va = f$ with V of the form

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

- Gaussian elimination ignores this structure and requires $O(n^3)$ operations to solve.

Polynomial Vandermonde Systems

- **Polynomial-Vandermonde** - Linear systems $V_P a = f$ with V_P of the form

$$V_P = \begin{bmatrix} p_0(x_1) & p_1(x_1) & p_2(x_1) & \cdots & p_{n-1}(x_1) \\ p_0(x_2) & p_1(x_2) & p_2(x_2) & \cdots & p_{n-1}(x_2) \\ p_0(x_3) & p_1(x_3) & p_2(x_3) & \cdots & p_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_0(x_n) & p_1(x_n) & p_2(x_n) & \cdots & p_{n-1}(x_n) \end{bmatrix}$$

  defined by a system of polynomials $P = \{p_0(x), p_1(x), \ldots, p_{n-1}(x)\}$ satisfying $\deg(p_k) = k$.

- Gaussian elimination again requires $O(n^3)$ operations to solve.

Numerical Illustrations - Experiment 1

Ill-Conditioned Matrices

#	cond(V)	GBP	GBP-IS	GE2t	GE3t
1	7e+14	5e-15	9e-15	8e-06	1e-05
2	1e+15	2e-15	2e-15	5e-05	8e-05
3	3e+15	3e-15	4e-15	6e-04	2e-04
4	1e+18	2e-15	2e-15	1e-01	7e-01
5	2e+15	4e-15	1e-15	4e-04	4e-04
6	5e+17	1e-14	1e-14	5e-02	3e-02
7	1e+16	4e-15	2e-15	2e-05	4e-05
8	1e+18	1e-15	1e-15	2e-02	1e-02
9	1e+18	2e-15	1e-15	1e-00	1e-00
10	9e+18	6e-16	8e-16	5e-01	7e-01

Numerical Illustrations - Experiment 2

Better Conditioned Matrices

#	cond(V)	GBP	GBP-IS	GE2t	GE3t
1	4e+08	3e-11	4e-11	1e-12	3e-12
2	2e+07	8e-11	1e-10	2e-14	5e-14
3	9e+05	1e-09	1e-09	9e-15	3e-14
4	3e+08	3e-11	3e-11	2e-13	2e-13
5	6e+09	2e-09	2e-09	1e-13	3e-13
6	1e+09	3e-11	1e-10	3e-14	8e-15
7	5e+05	3e-11	4e-11	1e-14	3e-14
8	3e+06	7e-11	9e-11	1e-13	7e-14
9	7e+05	1e-08	1e-08	2e-14	6e-14
10	2e+06	1e-10	3e-10	1e-13	1e-13

Definition

- A matrix $\mathbf{A}$ is **quasiseparable of order one** if

$$\max \text{Rank} A_{12} = \max \text{Rank} A_{21} = 1$$

where the maxima are taken over all symmetric partitions of the form

$$\mathbf{A} = \left[\begin{array}{c|c} * & \mathbf{A_{12}} \\ \hline \mathbf{A_{21}} & * \end{array}\right]$$

Proof

- For the **PN Sequence** listed above

$$\boxed{100011110101100}\ \boxed{100011110101100}\ \boxed{100011110101100}\ \ldots$$

$$T = \left[\begin{array}{c|cccc|ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}\right]$$

these $m = 4$ columns form a basis of the column space of $\widetilde{T}$

Observation: the PN matrix has the rank m

- **rank**$(\tilde{T}) = m$

A Decomposition of PN Matrices

$$\tilde{T} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \begin{bmatrix} I & R \end{bmatrix}$$

where $\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ is a $2^m - 1$ x m matrix, and $\begin{bmatrix} I & R \end{bmatrix}$ is an m x $2^m - 1$ matrix.

- Uniqueness of rows of $\tilde{T}$ imply uniqueness of rows in $\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$
- Uniqueness of columns of $\tilde{T}$ imply uniqueness of columns in $\begin{bmatrix} I & R \end{bmatrix}$
- This uniqueness and the sizes of these matrices imply they contain **all possible binary m-tuples** as rows and columns.

A Decomposition of Hadamard-Sylvester Matrices

- Let H'_n denote the Hadamard-Sylvester matrix H_n with $(1,-1) \rightarrow (0,1)$.
- Then H'_n has the decomposition

$$H'_n = \mathbf{L_m L_m^T}$$

where $L_m = [l_{ij}]$ is an n x m matrix with $l_{ij} \in \{0,1\}$ and the rows of L_m corresponding to all possible binary m-tuples.

Proof

This decomposition can be seen inductively:

- For $H'_2 = L_1 L_1^T$, we have $L_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

- Assuming $H'_m = L_m L_m^T$ is true for size m one sees that

$$H'_{(m+1)} = \begin{bmatrix} H'_m & H'_m \\ H'_m & \text{"}-\text{"}H'_m \end{bmatrix} = \underbrace{\begin{bmatrix} 0_m & \mathbf{L_m} \\ 1_m & \mathbf{L_m} \end{bmatrix}}_{L_{m+1}} \underbrace{\begin{bmatrix} {0_m}^T & {1_m}^T \\ \mathbf{L_m}^T & \mathbf{L_m}^T \end{bmatrix}}_{L_{m+1}^T}$$

where $0_m = \underbrace{[0 \; 0 \; \ldots \; 0]^T}_{m \text{ zeros}}$ $\quad 1_m = \underbrace{[1 \; 1 \; \ldots \; 1]^T}_{m \text{ ones}}$

This demonstrates the equivalence!