

Fast algorithms for polynomial–Vandermonde matrices related to quasiseparable matrices

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Outline

Fast algorithms for Vandermonde matrices

- ▶▶▶ Classical Björck-Pereyra & Traub algorithms
- ▶▶▶ Polynomial–Vandermonde matrices and previous work

Björck–Pereyra–like algorithm for quasiseparable–Vandermonde matrices

- ▶▶▶ Quasiseparable matrices & polynomials
- ▶▶▶ Björck–Pereyra–like algorithm

Efficient recurrence relations for quasiseparable polynomials

Traub–like algorithm for quasiseparable–Vandermonde matrices

Introduction

Toeplitz matrices and algorithms

▣▣▣▣ **Definition.** A **Toeplitz matrix** is defined by

$$t = \{t_{-n+1}, t_{-n+2}, \dots, t_0, \dots, t_{n-2}, t_{n-1}\}$$

$$\Downarrow$$

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-n+1} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-n+2} \\ t_2 & t_1 & t_0 & \cdots & t_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix}$$

Algorithms available for Toeplitz matrices

▣▣▣▣ Schur algorithm

▣▣▣▣ Levinson algorithm

▣▣▣▣ Transformation of Toeplitz to Cauchy-like ($C = F^*TF$)

Introduction

Vandermonde matrices and algorithms

►►► **Definition.** For a set of nodes, a **Vandermonde matrix** is defined by

$$x = \{x_1, x_2, \dots, x_n\}$$
$$\Downarrow$$
$$V(x) = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

Algorithms available for Vandermonde matrices

►►► Björck-Pereyra algorithm

►►► Traub algorithm

►►► Transformation of Vandermonde to Cauchy ($V(x) = D_1 C(x, y) D_2 V(y)$)

Fast Björck-Pereyra algorithm

► The **Björck-Pereyra algorithm** (1970) is based on the formula

$$V(x)^{-1} = U_1^{-1} \dots U_{n-1}^{-1} L_{n-1}^{-1} \dots L_1^{-1},$$

with

$$U_k^{-1} = \left[\begin{array}{c|ccc} I_{k-1} & & & \\ \hline & 1 & -x_k & \\ & & 1 & \ddots \\ & & & \ddots & -x_k \\ & & & & 1 \end{array} \right]$$

$$L_k^{-1} = \left[\begin{array}{c|ccc} I_{k-1} & & & \\ \hline & 1 & & \\ & & \frac{1}{x_{k+1}-x_k} & \\ & & & \ddots \\ & & & & \frac{1}{x_n-x_k} \end{array} \right] \left[\begin{array}{c|ccc} I_{k-1} & & & \\ \hline & 1 & & \\ & -1 & 1 & \\ & \vdots & & \ddots \\ & -1 & & & 1 \end{array} \right]$$

Fast Björck-Pereyra algorithm

- ▣ The solution a of the linear system

$$V(x)a = f$$

is computed by Björck-Pereyra as

$$a = V(x)^{-1}f = U_1^{-1} \dots U_{n-1}^{-1} L_{n-1}^{-1} \dots L_1^{-1} f$$

- ▣ Each matrix in the factorization is **sparse**, and so each matrix-vector product can be computed in $\mathcal{O}(n)$ operations.
- ▣ **Björck-Pereyra** requires only $\mathcal{O}(n^2)$ arithmetic operations vs $\mathcal{O}(n^3)$ of Gaussian elimination.

Fast Traub algorithm

► The **Traub algorithm** (1966) is based on the formula

$$V(x)^{-1} = \tilde{I} \cdot \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}^T \cdot \text{diag}(c_1, c_2, \dots, c_n)$$

where \tilde{I} is the antidiagonal matrix, $c_k = \prod_{\substack{j=1 \\ j \neq k}}^n (x_j - x_k)^{-1}$

► The polynomials $\{r_0(x), \dots, r_{n-1}(x)\}$ are the **Horner (associated) polynomials**, and satisfy **two-term recurrence relations**

$$r_0(x) = P_n, \quad r_k(x) = xr_{k-1}(x) + P_{n-k}$$

► **Fast:** requires only $\mathcal{O}(n^2)$ arithmetic operations vs $\mathcal{O}(n^3)$ of Gaussian elimination.

Conditioning of Vandermonde matrices

- ▶▶▶ The condition numbers of Vandermonde matrices **grow exponentially with their size** (Tyrtysnikov (1994)).
- ▶▶▶ Björck-Pereyra (1970) : “... *some problems, connected with Vandermonde systems, which traditionally have been considered to be too ill-conditioned to be attacked, actually can be solved with good precision*”.

Some Numerical Experiments

▣▣▣▣ Björck-Pereyra algorithm. (Higham's example)

1. Nodes chosen randomly in $(0, 1)$.

2. RHS alternating signs, $\begin{bmatrix} 1 & -1 & 1 & \cdots \end{bmatrix}^T$

3. Forward error measured by

$$e = \frac{\|x - \hat{x}\|_2}{\|x\|_2}$$

▣▣▣▣ Traub algorithm.

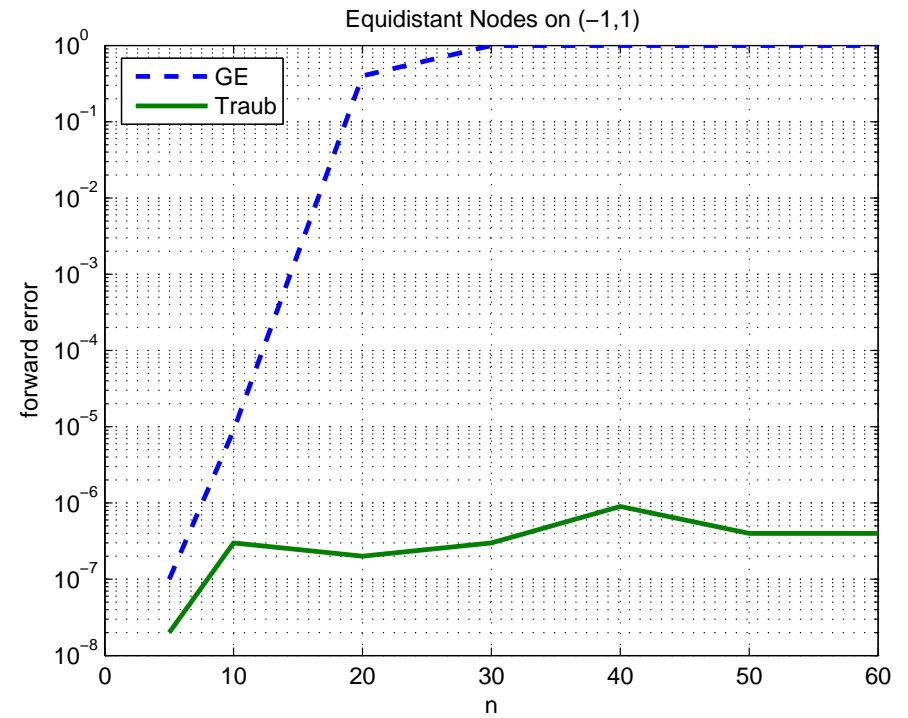
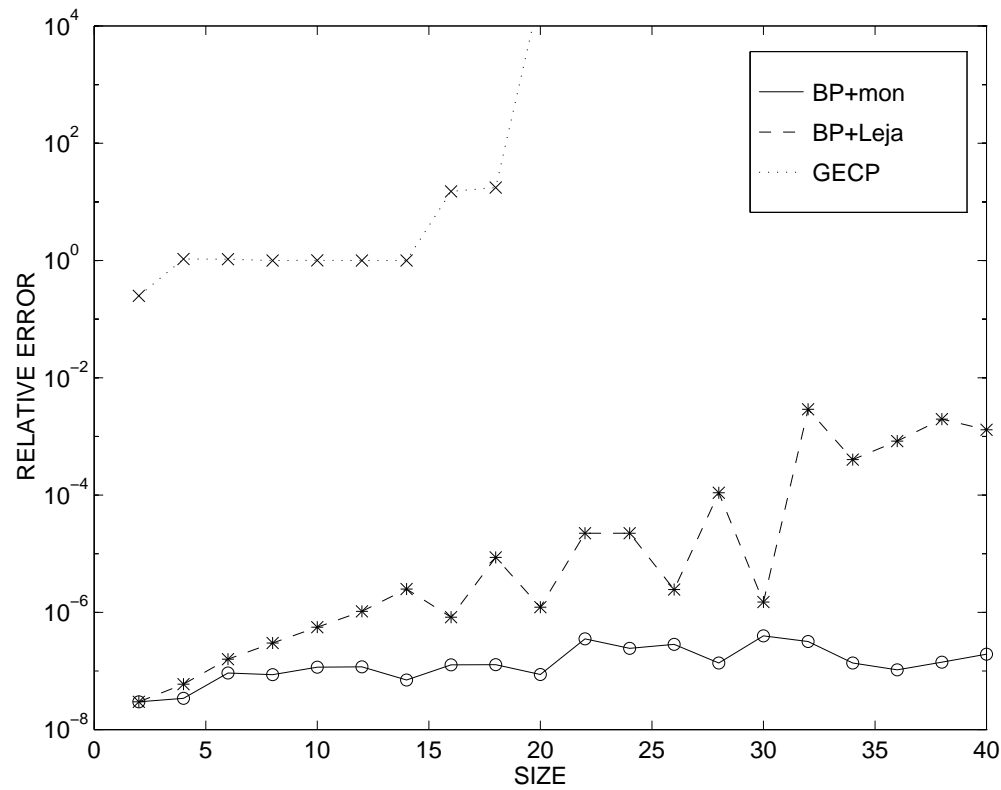
1. Nodes chosen randomly in $(0, 1)$.

2. Forward error measured by

$$e = \frac{\|A^{-1} - \widehat{A^{-1}}\|_2}{\|A^{-1}\|_2}$$

Numerical Experiments

Classical Björck-Pereyra & Traub Algorithms



Error Analysis

Björck-Pereyra Algorithm

- ▣▣▣▣ **Higham** (1990) showed that under some conditions (sign-oscillating right-hand-side and monotonic ordering of the nodes, all of which are positive), there is a **provably excellent forward error bound**:

$$\frac{|a - \hat{a}|}{|a|} \leq 5nu + \mathcal{O}(u^2)$$

- ▣▣▣▣ **Conclusion:** Very good forward error is to be expected under some conditions.

Generalizations of these algorithms

Polynomial–Vandermonde matrices

► **Definition.** For sets of polynomials and nodes, define a **polynomial–Vandermonde matrix**:

$$\begin{aligned}
 x &= \{x_1, x_2, \dots, x_n\} \\
 R &= \{r_0(x), r_1(x), \dots, r_{n-1}(x)\} \\
 &\quad \Downarrow \\
 V_R(x) &= \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}
 \end{aligned}$$

► **Note.** If the polynomial system R satisfies nice recurrence relations, then the n^2 entries of $V_R(x)$ can be defined by only $\mathcal{O}(n)$ parameters.

Fast algorithms for polynomial–Vandermonde matrices

Previous work

| polynomial–Vandermonde matrix | Björck-Pereyra-type | Traub-type |
|-------------------------------|----------------------|--------------------------|
| Vandermonde | Björck-Pereyra(1970) | Traub (1966) |
| Chebyshev-Vandermonde | Reichel-Opfer (1991) | Gohberg-Olshevsky (1994) |
| three-term Vandermonde | Higham (1988,90) | Calvetti-Reichel (1993) |
| Szegö-Vandermonde | BEGKO (2006) | Olshevsky (2001) |

Main results

- ▣▶ A generalization of the Björck-Pereyra algorithm that includes all of these previous cases.
- ▣▶ A generalization of the Traub algorithm that includes all of these previous cases.

How?

Fast algorithms for polynomial–Vandermonde matrices

Previous work

| polynomial–Vandermonde matrix | Björck-Pereyra-type | Traub-type |
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| Vandermonde | Björck-Pereyra(1970) | Traub (1966) |
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| three-term Vandermonde | Higham (1988,90) | Calvetti-Reichel (1993) |
| Szegö-Vandermonde | BEGKO (2006) | Olshevsky (2001) |
| Quasiseparable-Vandermonde | ???????????????? | ???????????????? |

Main results

- ▣▶ A generalization of the Björck-Pereyra algorithm that includes all of these previous cases.
- ▣▶ A generalization of the Traub algorithm that includes all of these previous cases.

QUASISEPARABLE MATRICES

Quasiseparable Matrices

⇒ **Definition.** A matrix C is $(H, 1)$ -**quasiseparable** if it is upper Hessenberg and

$$\max \text{Rank} C_{12} = 1$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline * & * \end{array} \right]$$

⇒ **Previous Work.** Gohberg-Kaashoek-Lerer, Dewilde, Gohberg-Eidelman, Van Barel et al, Tyrtyshnikov et al, Bini et al, Gu-Chandrasekaran et al.

Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

► The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C is the **monomials** with recurrence relations

$$r_k(x) = xr_{k-1}(x)$$

► The matrix C is $(H, 1)$ -**quasiseparable**.

Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C are **real orthogonal polynomials** with recurrence relations

$$r_k(x) = \frac{1}{q_k} (x - d_k) r_{k-1}(x) - \frac{g_{k-1}}{q_k} r_{k-2}(x)$$

- The matrix C is **$(H, 1)$ -quasiseparable**.

Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

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Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

► The system of polynomials $r_k(x) = \det(xI - C_{k \times k})$ associated with C are the **Szegő polynomials** with recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ xr_{k-1}(x) \end{bmatrix}$$

► The matrix C is $(H, 1)$ -**quasiseparable**.

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

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Important Special Cases

Unitary Hessenberg

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A generator representation

⇒ The following matrix is $(H, 1)$ -**quasiseparable**:

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

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 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\
 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\
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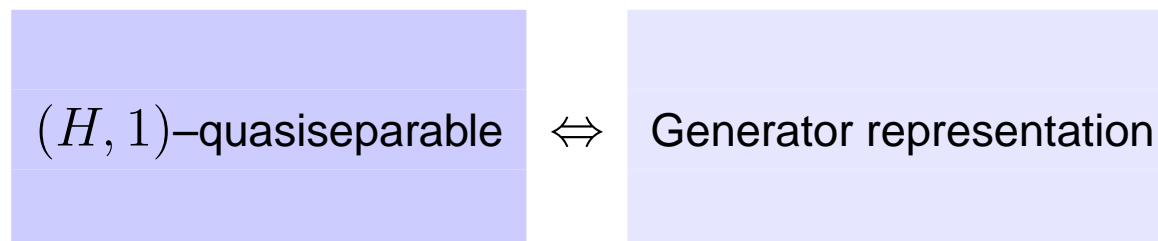
$$\begin{bmatrix}
 d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\
 p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\
 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\
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 \end{bmatrix}$$

A generator representation

► The following matrix is $(H, 1)$ -**quasiseparable**:

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

► This **generator representation** exists for any $(H, 1)$ -**quasiseparable** matrix.



Quasiseparable-Vandermonde matrices

▣▣▣▣ **Definition.** A **Quasiseparable-Vandermonde matrix** is of the form

$$V_R = \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}$$

where the polynomials $r_k(x)$ defined by $r_k(x) = \det(xI - C_{k \times k})$ correspond to an $(H, 1)$ -**quasiseparable** matrix C .

▣▣▣▣ The class of $(H, 1)$ -**quasiseparable** polynomials contains as subclasses the classes of **real-orthogonal polynomials** and **Szegő polynomials**.

A Björck–Pereyra–like algorithm for quasiseparable–Vandermonde matrices

Joint work with Yuli Eidelman, Israel Gohberg, Israel Koltracht, & Vadim Olshevsky

Björck–Pereyra–like algorithm for quasiseparable–Vandermonde matrices

Like the Björck–Pereyra algorithm, the generalization is based on the formula

$$V_R^{-1} = U_1^{-1} \cdots U_{n-1}^{-1} L_{n-1}^{-1} \cdots L_1^{-1},$$

with

$$U_k^{-1} = \text{diag} \left\{ I_{k-1}, \begin{bmatrix} \frac{1}{\alpha_0} & & & \\ 0 & \boxed{C - x_k I} & & \\ \vdots & & \ddots & \\ 0 & & & \frac{1}{\alpha_{n-k}} \end{bmatrix} \right\}$$

$$L_k^{-1} = \begin{bmatrix} I_{k-1} & & & & \\ \hline & 1 & & & \\ & & \frac{1}{x_{k+1} - x_k} & & \\ & & & \ddots & \\ & & & & \frac{1}{x_n - x_k} \end{bmatrix} \begin{bmatrix} I_{k-1} & & & & \\ \hline & 1 & & & \\ & -1 & 1 & & \\ & \vdots & & \ddots & \\ & -1 & & & 1 \end{bmatrix}$$

Complexity of the Björck–Pereyra–like algorithm

- ▶ To design a fast algorithm, we need **fast multiplication of C by a vector**.
The **classical Björck–Pereyra algorithm** (monomial case), C is **bidiagonal**.
In the **Szegö** case, C is **unitary Hessenberg**, and hence admits a convenient **Schur factorization**.
- ▶ How does one multiply a $(H, 1)$ –**quasiseparable** matrix by a vector in $\mathcal{O}(n)$ operations?

Complexity of the Björck–Pereyra–like algorithm

Proposition. An $(H, 1)$ –quasiseparable matrix C admits the decomposition

$$C = L + U$$

for a lower–bidiagonal matrix L and

$$U = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & g_{n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & & & \\ \vdots & \tilde{B}^{-1} & & \\ 0 & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h_n \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} I & -b_2 & & \\ & \ddots & \ddots & \\ & & I & -b_{n-1} \\ & & & I \end{bmatrix}$$

Complexity of the Björck–Pereyra–like algorithm

- ▶▶▶ Thus, in the **quasiseparable** case, C can be multiplied by a vector at the cost of a multiplication by a bidiagonal matrix, two diagonal scalings, and a back–substitution with a bidiagonal matrix.
- ▶▶▶ This leads to an $\mathcal{O}(n)$ algorithm.
- ▶▶▶ This implementation coincides with the algorithm derived differently by **Eidelman and Gohberg** (1999).
- ▶▶▶ Thus the cost of the Björck–Pereyra–like algorithm is $\mathcal{O}(n^2)$ arithmetic operations.

Special cases of the new Björck–Pereyra–like algorithm

- ▶▶▶ **Tridiagonal** case: This algorithm reduces to the **Higham algorithm** (1990).
 - **Stage I** corresponds to computing $L_{n-1}^{-1} \dots L_1^{-1} f = x$.
 - **Stage II** corresponds to computing $U_1^{-1} \dots U_{n-1}^{-1} x$.
- ▶▶▶ **Unitary Hessenberg** case: This reduces to the **BEGKO algorithm** (2006).

Numerical Illustrations - Björck-Pereyra

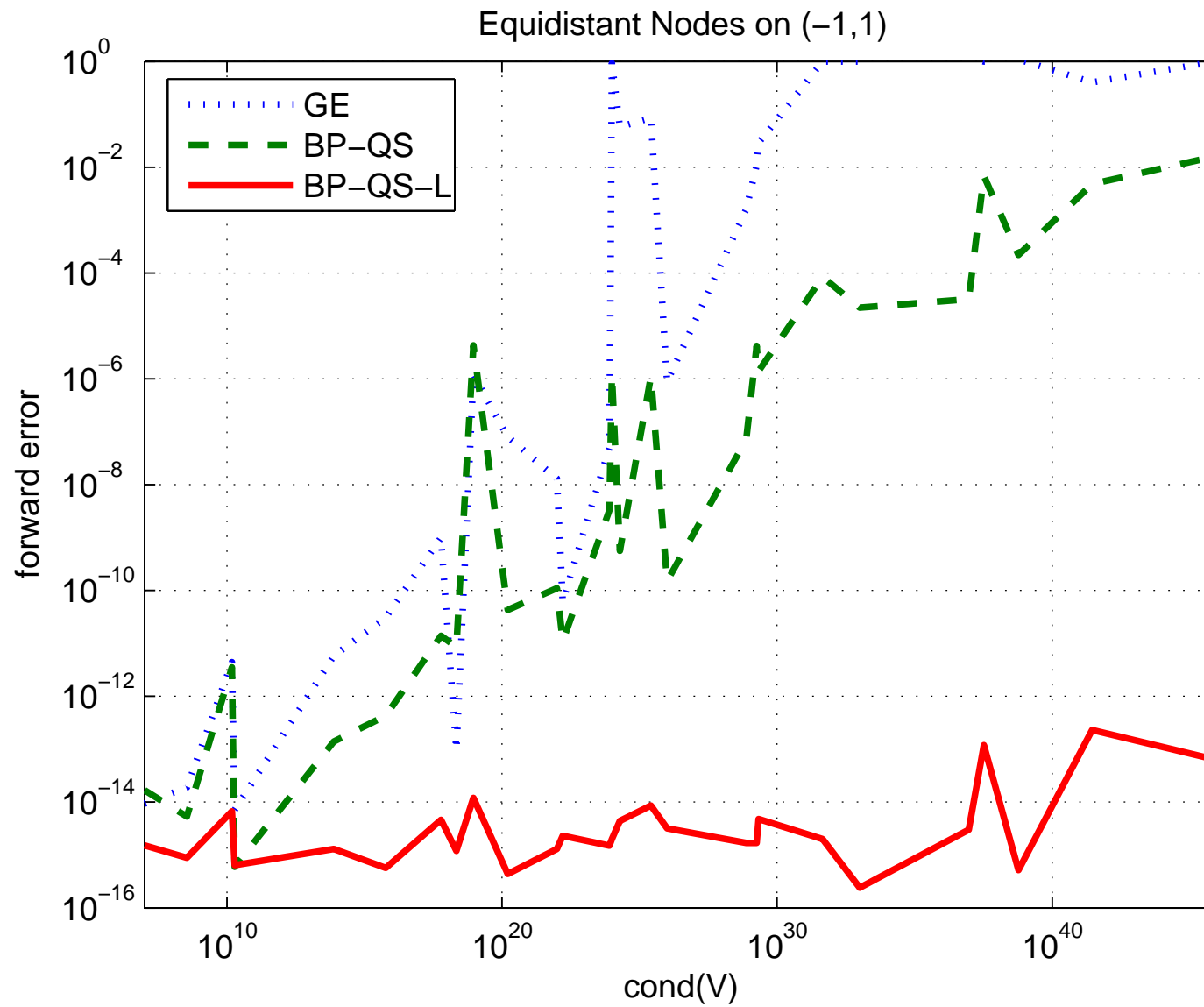
- ▶▶▶ We compare the **forward error** of the solutions \hat{x} from MATLAB in double precision via

$$e = \frac{\|x - \hat{x}\|_2}{\|x\|_2},$$

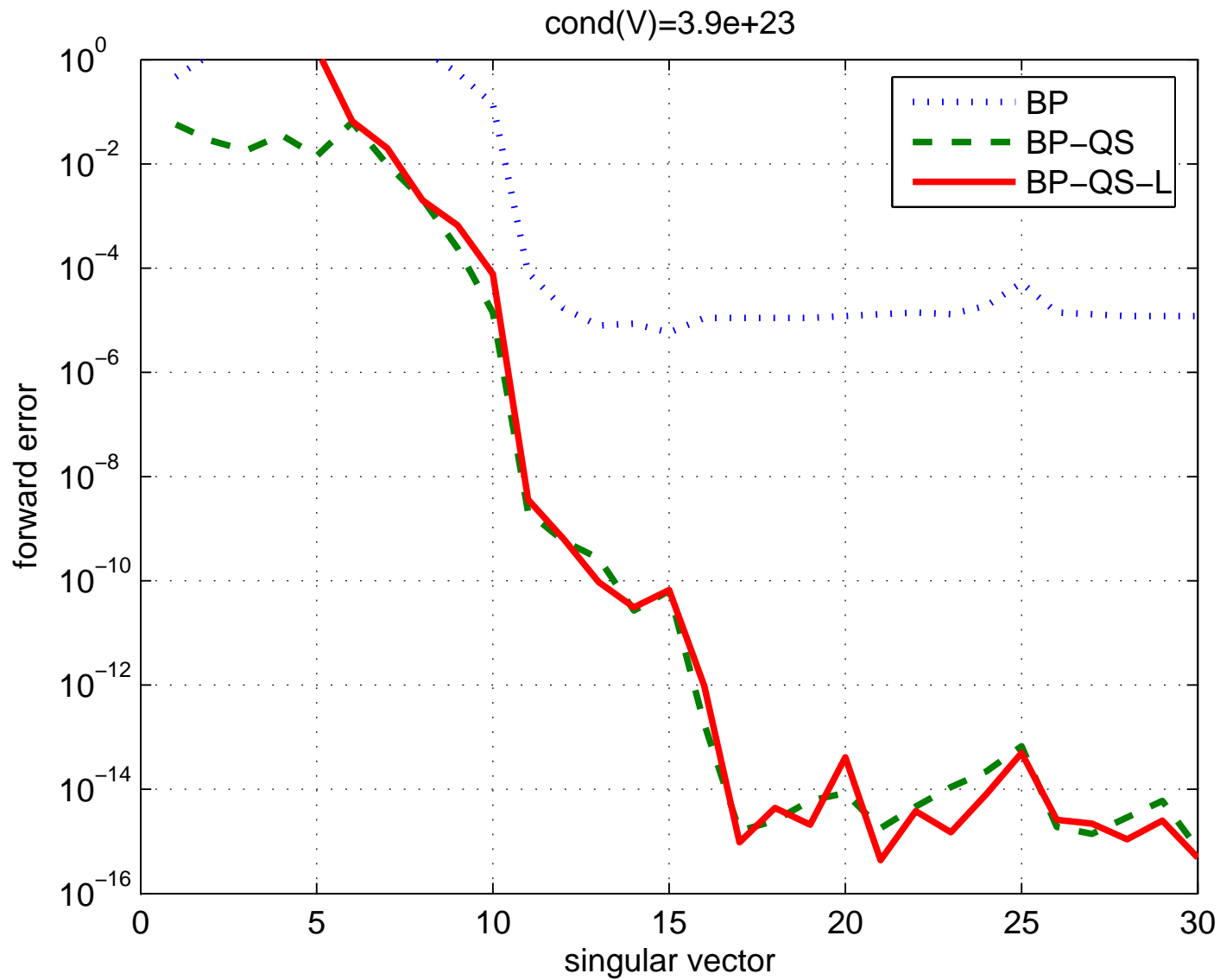
with x , the “exact” solution using MATLAB’s `vpa ()` command for software-implemented arbitrary digit arithmetic.

- ▶▶▶ **GE** - Gaussian elimination via MATLAB’s backslash command.
- ▶▶▶ **BP-QS** - Björck–Pereyra–like algorithm.
- ▶▶▶ **BP-QS-L** - Björck–Pereyra–like algorithm with nodes ordered via the **Leja ordering**. (Reichel, Higham)

Numerical Illustrations - BP Experiment 1



Numerical Illustrations - BP Experiment 2



Fast algorithms for polynomial–Vandermonde matrices

Previous work

| polynomial–Vandermonde matrix | Björck-Pereyra-type | Traub-type |
|-------------------------------|----------------------|--------------------------|
| Vandermonde | Björck-Pereyra(1970) | Traub (1966) |
| Chebyshev-Vandermonde | Reichel-Opfer (1991) | Gohberg-Olshevsky (1994) |
| three-term Vandermonde | Higham (1988,90) | Calvetti-Reichel (1993) |
| Szegö-Vandermonde | BEGKO (2006) | Olshevsky (2001) |
| Quasiseparable-Vandermonde | BEGKO (2007) | ???????????????? |

Efficient recurrence relations for quasiseparable polynomials

| polynomial–Vandermonde matrix | Polynomials $r_k(x)$ |
|-------------------------------|-----------------------------|
| Vandermonde | Monomials |
| Chebyshev-Vandermonde | Chebyshev polynomials |
| three-term Vandermonde | Real-orthogonal polynomials |
| Szegö-Vandermonde | Szegö polynomials |
| Quasiseparable-Vandermonde | Quasiseparable polynomials |

$$V_R(x) = \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}$$

Efficient recurrence relations for quasiseparable polynomials

| | |
|-------------------------------|-----------------------------|
| polynomial–Vandermonde matrix | Polynomials $r_k(x)$ |
| Vandermonde | Monomials |
| Chebyshev–Vandermonde | Chebyshev polynomials |
| three-term Vandermonde | Real–orthogonal polynomials |
| Szegö–Vandermonde | Szegö polynomials |
| Quasiseparable–Vandermonde | Quasiseparable polynomials |

Recurrence relations

$$r_k(x) = xr_{k-1}(x)$$

Efficient recurrence relations for quasiseparable polynomials

| | |
|-------------------------------|------------------------------|
| polynomial–Vandermonde matrix | Polynomials $r_k(x)$ |
| Vandermonde | Monomials |
| Chebyshev-Vandermonde | Chebyshev polynomials |
| three-term Vandermonde | Real–orthogonal polynomials |
| Szegö-Vandermonde | Szegö polynomials |
| Quasiseparable-Vandermonde | Quasiseparable polynomials |

Recurrence relations

$$r_k(x) = 2xr_{k-1}(x) - r_{k-2}(x)$$

Efficient recurrence relations for quasiseparable polynomials

| | |
|-------------------------------|-----------------------------|
| polynomial–Vandermonde matrix | Polynomials $r_k(x)$ |
| Vandermonde | Monomials |
| Chebyshev-Vandermonde | Chebyshev polynomials |
| three-term Vandermonde | Real–orthogonal polynomials |
| Szegö-Vandermonde | Szegö polynomials |
| Quasiseparable-Vandermonde | Quasiseparable polynomials |

Recurrence relations

$$r_k(x) = \frac{1}{q_k}(x - d_k)r_{k-1}(x) - \frac{g_{k-1}}{q_k}r_{k-2}(x)$$

Efficient recurrence relations for quasiseparable polynomials

| | |
|-------------------------------|-----------------------------|
| polynomial–Vandermonde matrix | Polynomials $r_k(x)$ |
| Vandermonde | Monomials |
| Chebyshev-Vandermonde | Chebyshev polynomials |
| three-term Vandermonde | Real–orthogonal polynomials |
| Szegö-Vandermonde | Szegö polynomials |
| Quasiseparable-Vandermonde | Quasiseparable polynomials |

Recurrence relations (2-term)

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ xr_{k-1}(x) \end{bmatrix}$$

Efficient recurrence relations for quasiseparable polynomials

| | |
|-------------------------------|-----------------------------|
| polynomial–Vandermonde matrix | Polynomials $r_k(x)$ |
| Vandermonde | Monomials |
| Chebyshev-Vandermonde | Chebyshev polynomials |
| three-term Vandermonde | Real–orthogonal polynomials |
| Szegö-Vandermonde | Szegö polynomials |
| Quasiseparable-Vandermonde | Quasiseparable polynomials |

Recurrence relations (3-term)

$$r_k(x) = -\frac{1}{\mu_k} \left(x + \frac{\rho_k}{\rho_{k-1}} \right) r_{k-1}(x) - \left(\frac{\rho_k \mu_{k-1}}{\rho_{k-1} \mu_k} \right) x \cdot r_{k-2}(x)$$

Efficient recurrence relations for quasiseparable polynomials

| | |
|-------------------------------|-----------------------------|
| polynomial–Vandermonde matrix | Polynomials $r_k(x)$ |
| Vandermonde | Monomials |
| Chebyshev–Vandermonde | Chebyshev polynomials |
| three-term Vandermonde | Real–orthogonal polynomials |
| Szegö–Vandermonde | Szegö polynomials |
| Quasiseparable–Vandermonde | Quasiseparable polynomials |

Recurrence relations

????????????????????

Three special cases of recurrence relations for quasiseparable polynomials

Szegő-type two-term recurrence relations

Theorem. A system of quasiseparable polynomials satisfies recurrence relations of the form

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

provided the generators of the matrix satisfy

$$b_k \neq 0, \quad k = 2, 3, \dots, n - 1$$

► Further, the exact class of polynomials for which these recurrence relations are applicable is the polynomials related to Hessenberg **order-one semiseparable** matrices.

Three special cases of recurrence relations for quasiseparable polynomials

General three-term recurrence relations

Theorem. A system of quasiseparable polynomials satisfies recurrence relations of the form

$$\begin{aligned}r_0(x) &= 1, & r_1(x) &= (\alpha_1 x - \delta_1) \cdot r_0(x) \\ r_k(x) &= (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)\end{aligned}$$

provided the generators of the matrix satisfy

$$h_k \neq 0, \quad k = 2, 3, \dots, n$$

► Further, the exact class of polynomials for which these recurrence relations are applicable is the polynomials related to what we suggest to call **well-free** matrices.

Three special cases of recurrence relations for quasiseparable polynomials

[EGO05]–type two–term recurrence relations

Theorem. A system of quasiseparable polynomials satisfies recurrence relations of the form

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

where the generators of the matrix are under

NO RESTRICTIONS.

Special cases of these recurrence relations

| Matrix | Polynomial system |
|---------------------------|-----------------------------|
| Shift matrix | monomials |
| Tridiagonal matrix | real orthogonal polynomials |
| Unitary Hessenberg matrix | Szegő polynomials |

Special cases of these recurrence relations

| Matrix | Polynomial system |
|---------------------------|-----------------------------|
| Shift matrix | monomials |
| Tridiagonal matrix | real orthogonal polynomials |
| Unitary Hessenberg matrix | Szegő polynomials |

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

$$\Downarrow$$

$$p_k = 1, \quad q_k = 1, \quad d_k = 0, \quad g_k = 0, \quad b_k = 0, \quad h_k = 1$$

$$\Downarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Special cases of these recurrence relations

General three-term recurrence relations in the monomial case

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)$$

⇓

$$p_k = 1, \quad q_k = 1, \quad d_k = 0, \quad g_k = 0, \quad b_k = 0, \quad h_k = 1$$

⇓

$$r_k(x) = x \cdot r_{k-1}(x)$$

Special cases of these recurrence relations

| Matrix | Polynomial system |
|---------------------------|-----------------------------|
| Shift matrix | monomials |
| Tridiagonal matrix | real orthogonal polynomials |
| Unitary Hessenberg matrix | Szegő polynomials |

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

 \Downarrow

$$p_k = 1, \quad b_k = 0, \quad h_k = 1$$

 \Downarrow

$$\begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

Special cases of these recurrence relations

General three-term recurrence relations in the real orthogonal polynomial case

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)$$

⇓

$$p_k = 1, \quad b_k = 0, \quad h_k = 1$$

⇓

$$r_k(x) = \frac{1}{q_k} (x - d_k) r_{k-1}(x) - \frac{g_{k-1}}{q_k} r_{k-2}(x)$$

Special cases of these recurrence relations

| Matrix | Polynomial system |
|---------------------------|-----------------------------|
| Shift matrix | monomials |
| Tridiagonal matrix | real orthogonal polynomials |
| Unitary Hessenberg matrix | Szegő polynomials |

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ 0 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ 0 & 0 & p_4 q_3 & d_4 & g_4 h_5 \\ 0 & 0 & 0 & p_5 q_4 & d_5 \end{bmatrix}$$

⇓

$$p_k = 1, \quad q_k = \mu_k, \quad d_k = -\rho_{k-1}^* \rho_k, \quad g_k = \rho_{k-1}^*, \quad b_k = \mu_{k-1}, \quad h_k = -\mu_{k-1} \rho_k$$

⇓

$$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & -\rho_0^* \mu_1 \mu_2 \mu_3 \rho_4 & -\rho_0^* \mu_1 \mu_2 \mu_3 \mu_4 \rho_5 \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & -\rho_1^* \mu_2 \mu_3 \rho_4 & -\rho_1^* \mu_2 \mu_3 \mu_4 \rho_5 \\ 0 & \mu_2 & -\rho_2^* \rho_3 & -\rho_2^* \mu_3 \rho_4 & -\rho_2^* \mu_3 \mu_4 \rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^* \rho_4 & -\rho_3^* \mu_4 \rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^* \rho_5 \end{bmatrix}$$

Special cases of these recurrence relations

Szegő–type recurrence relations in the Szegő polynomial case

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

⇓

$$p_k = 1, \quad q_k = \mu_k, \quad d_k = -\rho_{k-1}^* \rho_k, \quad g_k = \rho_{k-1}^*, \quad b_k = \mu_{k-1}, \quad h_k = -\mu_{k-1} \rho_k$$

⇓

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ x r_{k-1}(x) \end{bmatrix}$$

Traub–like algorithm for quasiseparable-Vandermonde matrices

Joint work with Yuli Eidelman, Israel Gohberg, Vadim Olshevsky, & Eugene Tyrtyshnikov

Traub-like algorithm for quasiseparable-Vandermonde matrices

►►► **Traub-like algorithm.** Based on the formula

$$V_R^{-1} = \begin{bmatrix} \hat{r}_0(x_1) & \hat{r}_1(x_1) & \cdots & \hat{r}_{n-1}(x_1) \\ \hat{r}_0(x_2) & \hat{r}_1(x_2) & \cdots & \hat{r}_{n-1}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{r}_0(x_n) & \hat{r}_1(x_n) & \cdots & \hat{r}_{n-1}(x_n) \end{bmatrix}^{-1} = \tilde{I} \cdot V_{\hat{R}}^T \cdot \text{diag}(c_1, c_2, \dots, c_n)$$

where \tilde{I} is the antidiagonal matrix, $c_k = \prod_{\substack{j=1 \\ j \neq k}}^n (x_j - x_k)^{-1}$

- \hat{R} is the system of **Horner-like** polynomials corresponding to the polynomial system R .
 (When P is the monomial basis, this is the classical Traub (1966))
- **How do we evaluate the polynomials \hat{r}_k at the nodes?**

Szegő-type two-term recurrence relations

Original polynomials forming V_R

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k)r_{k-1}(x) \end{bmatrix}$$

Horner-like polynomials forming V_R^{-1}

$$\begin{bmatrix} \widehat{G}_k(x) \\ \widehat{r}_k(x) \end{bmatrix} = \begin{bmatrix} \widehat{\alpha}_k & \widehat{\beta}_k \\ \widehat{\gamma}_k & 1 \end{bmatrix} \begin{bmatrix} \widehat{G}_{k-1}(x) \\ (\widehat{\delta}_k x + \widehat{\theta}_k)\widehat{r}_{k-1}(x) + P_{n-k} \end{bmatrix}$$

These recurrence relations may be used provided the generators of the matrix satisfy

$$b_k \neq 0, \quad k = 2, 3, \dots, n - 1$$

General three-term recurrence relations

Original polynomials forming V_R

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x)$$

Horner-like polynomials forming V_R^{-1}

$$\begin{aligned} \hat{r}_k(x) &= (\hat{\alpha}_k x - \hat{\delta}_k) \cdot \hat{r}_{k-1}(x) - (\hat{\beta}_k x + \hat{\gamma}_k) \cdot \hat{r}_{k-2}(x) \\ &\quad + \hat{\alpha}_k P_{n-k} - \hat{\beta}_k P_{n-k+1} \end{aligned}$$

These recurrence relations may be used provided the generators of the matrix satisfy

$$g_k \neq 0, \quad k = 1, 2, \dots, n - 1$$

[EGO05]-type two-term recurrence relations**Original polynomials forming V_R**

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

Horner-like polynomials forming V_R^{-1}

$$\begin{bmatrix} \widehat{F}_k(x) \\ \widehat{r}_k(x) \end{bmatrix} = \begin{bmatrix} \widehat{\alpha}_k & \widehat{\beta}_k \\ \widehat{\gamma}_k & \widehat{\delta}_k x + \widehat{\theta}_k \end{bmatrix} \begin{bmatrix} \widehat{F}_{k-1}(x) \\ \widehat{r}_{k-1}(x) \end{bmatrix} + \widehat{\delta}_k \begin{bmatrix} 0 \\ P_{n-k} \end{bmatrix}$$

These recurrence relations may be used regardless of the generators of the matrix.

What are the coefficients P_k ?

- ▶▶▶ The difference between the recurrence relations for the original polynomials and those for the Horner-like polynomials is the presence of the coefficients P_k .
- ▶▶▶ P_k is the coefficient of $r_k(x)$ in the decomposition of the **master polynomial** $P(x) = \prod_{i=1}^n (x - x_i)$ into the $\{r_k\}$ basis:

$$\prod_{i=1}^n (x - x_i) = P_0 r_0(x) + P_1 r_1(x) + \cdots + P_n r_n(x)$$

- ▶▶▶ These coefficients can be computed in $\mathcal{O}(n^2)$ arithmetic operations.

Complexity of the Traub-like algorithm

- ▶▶▶ Each Horner-like polynomial can be evaluated at all of the nodes in $\mathcal{O}(n)$ operations.
- ▶▶▶ The coefficients P_k can be computed in $\mathcal{O}(n^2)$ operations.
- ▶▶▶ The total cost of the algorithm is $\mathcal{O}(n^2)$ operations. Comparing this to the complexity of Gaussian elimination, $\mathcal{O}(n^3)$, we have the algorithm is **FAST!**

Special cases of the Traub-like algorithm

- ▶▶▶ **Tridiagonal** case: Using the general three-term recurrence relations reduces to the algorithm of **Calvetti-Reichel** (1993).
- ▶▶▶ **Unitary Hessenberg** case: Using the Szegő-type two-term recurrence relations reduces to the algorithm of **Olshevsky** (2001).

Fast algorithms for polynomial-Vandermonde matrices

Previous work

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|-------------------------------|----------------------|--------------------------|
| Vandermonde | Björck-Pereyra(1970) | Traub (1966) |
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| three-term Vandermonde | Higham (1988,90) | Calvetti-Reichel (1993) |
| Szegö-Vandermonde | BEGKO (2006) | Olshevsky (2001) |
| Quasiseparable-Vandermonde | BEGKO (2007) | BEGOT (2007) |

Fast algorithms for polynomial–Vandermonde matrices related to quasiseparable matrices

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Joint work with:

Yuli Eidelman, Israel Gohberg, Israel Koltracht, Vadim Olshevsky, & Eugene Tyrtyshnikov.

Hadamard–Sylvester and Pseudo–noise matrices are equivalent

Joint work with Vadim Olshevsky & Lev Sakhnovich

▣▶ A **Hadamard matrix** is one whose entries are ± 1 and satisfy $H_n^T H_n = nI_n$.

A **Hadamard–Sylvester matrix** is a Hadamard matrix built from the recursion $H_1 = [1]$,

$$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}.$$

▣▶ The output of a linear–shift register corresponding to a primitive polynomial is called a **Pseudo–noise sequence**.

A **Pseudo–noise matrix** is a padded, circulant Hankel matrix whose rows are Pseudo–noise sequences.

▣▶ **Theorem.** Hadamard–Sylvester matrices and Pseudo–noise matrices are **equivalent**; i.e. one can be obtained from the other via row and column permutations.

Structure-preserving perturbations of matrices self-adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

- For a Hermitian, invertible (not necessarily positive definite) matrix H , one defines the **indefinite inner product** via

$$[x, y]_H = (Hx, y) = y^* Hx$$

where (\cdot, \cdot) denotes the Euclidean inner product.

- One defines **self-adjoint with respect to an indefinite inner product** in an analogous way to the definition for classical self-adjoint.

| Euclidean Inner Product | Indefinite Inner Product |
|-------------------------|--------------------------|
| $(Ax, y) = (x, Ay)$ | $[Ax, y]_H = [x, Ay]_H$ |
| $A = A^*$ | $HA = A^* H$ |

Structure-preserving perturbations of matrices self-adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

- ▣ Pairs of matrices (A, H) have a **canonical form** (J, P) [Gohberg–Lancaster–Rodman 1983], with

$$J = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus \tilde{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \tilde{J}(\lambda_\beta)$$

and

$$P = \epsilon_1 P_1 \oplus \cdots \oplus \epsilon_\alpha P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P(\lambda_\beta)$$

where $\lambda_1, \dots, \lambda_\alpha \in \mathbb{R}$, $J(\lambda)$ is a Jordan block, and $\tilde{J}(\lambda) = J(\lambda) \oplus J(\bar{\lambda})$.

- ▣ The matrix that reduces to this canonical form, T such that

$$T^{-1}AT = J, \quad T^*HT = P$$

has columns that are not only a Jordan basis of A , they also bring H into P , and we call such a basis a **canonical Jordan basis of** (A, H) .

Structure-preserving perturbations of matrices self-adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

► **Theorem.** Let $A_0 \in \mathbb{C}^{n \times n}$ be a fixed H_0 -selfadjoint matrix. Let

$$\left\{ \left\{ f_r^{(k,s)} \right\}_{r=0}^{m_k(A_0, \lambda_s) - 1} \right\}_{s=1, k=1}^{s=\beta, k=\dim \ker(A_0 - \lambda_s I)}$$

be a fixed canonical Jordan basis of A_0 . There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any H -selfadjoint matrix A such that A has the same Jordan structure as A_0 and

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists a canonical Jordan basis

$$\left\{ \left\{ g_r^{(k,s)} \right\}_{r=0}^{m_k(A, \lambda_s) - 1} \right\}_{s=1, k=1}^{s=\beta, k=\dim \ker(A - \lambda_s I)}$$

of A such that

$$\|g_r^{(k,s)} - f_r^{(k,s)}\| \leq K (\|A - A_0\| + \|H - H_0\|)$$

for all k, s, r within their ranges.

Fast algorithms for polynomial–Vandermonde matrices related to quasiseparable matrices

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Joint work with:

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Supplemental Slides

Confederate Matrices

► **Definition** For polynomials $R = \{r_0(x), r_1(x), \dots, r_n(x)\}$ satisfying n -term recurrence relations

$$r_k(x) = \alpha_k \cdot x r_{k-1}(x) - a_{k-1,k} \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \dots - a_{0,k} \cdot r_0(x).$$

and the polynomial

$$b(x) = b_0 \cdot r_0(x) + b_1 \cdot r_1(x) + \dots + b_{n-1} \cdot r_{n-1}(x) + b_n \cdot r_n(x)$$

define the **confederate matrix** of b with respect to R by

$$C_R(b) = \begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \dots & \dots & \frac{a_{0,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_0}{b_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \dots & \dots & \frac{a_{1,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_1}{b_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \dots & \dots & \frac{a_{2,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_2}{b_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & \dots & \dots & & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \dots & & & & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_{n-1}}{b_n} \end{bmatrix}$$

Confederate Matrices

$$r_1(x) = \alpha_1 \cdot xr_0(x) - a_{0,1} \cdot r_0(x)$$

$$r_2(x) = \alpha_2 \cdot xr_1(x) - a_{1,2} \cdot r_1(x) - a_{0,2} \cdot r_0(x)$$

$$r_3(x) = \alpha_3 \cdot xr_2(x) - a_{2,3} \cdot r_2(x) - a_{1,3} \cdot r_1(x) - a_{0,3} \cdot r_0(x)$$

$$\vdots$$

$$\left[\begin{array}{cccccc} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n} \end{array} \right]$$

Confederate Matrices

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$$\left[\begin{array}{cccccc} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n} \end{array} \right]$$

Confederate Matrices

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$$\begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_0}{b_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_1}{b_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_2}{b_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & & & & & \vdots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & \vdots & \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} & - & \frac{1}{\alpha_n} & \cdot & \frac{b_{n-1}}{b_n} \end{bmatrix}$$

Motivation for Horner-like Polynomials

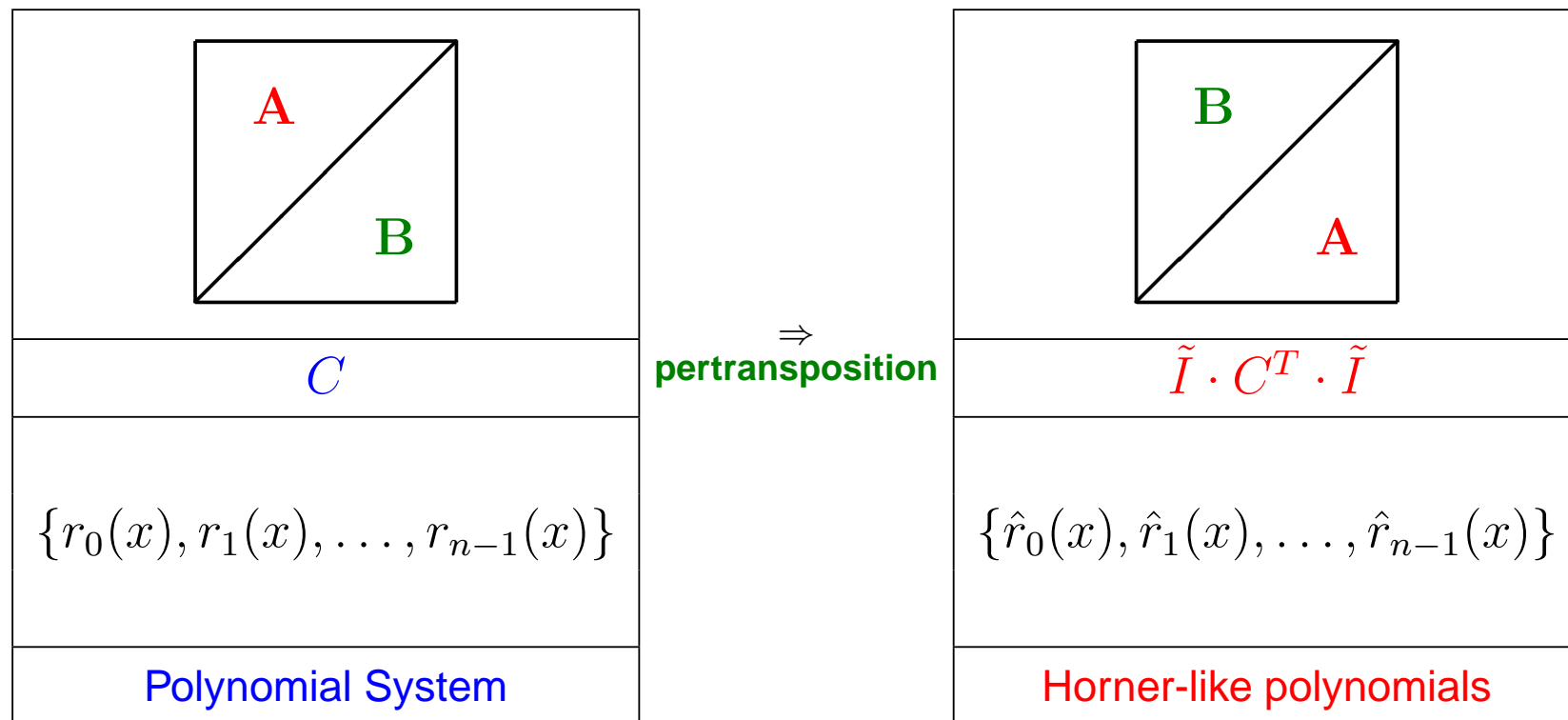
- The confederate matrix $C(b)$ for a polynomial $b(x) = b_0 + b_1x + \dots + b_nx_n$ in the **monomial basis** reduces to the companion matrix, and the confederate matrix $C_R(\hat{p}_n)$ for the **Horner polynomials** is:

$$\mathbf{C}(\mathbf{b}) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -b_{n-1} \end{bmatrix} \quad \mathbf{C}_R(\hat{\mathbf{p}}_n) = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -b_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

- **Observation:** $\mathbf{C}_R(\hat{\mathbf{p}}_n) = \tilde{I} \cdot \mathbf{C}(\mathbf{b})^T \cdot \tilde{I}$

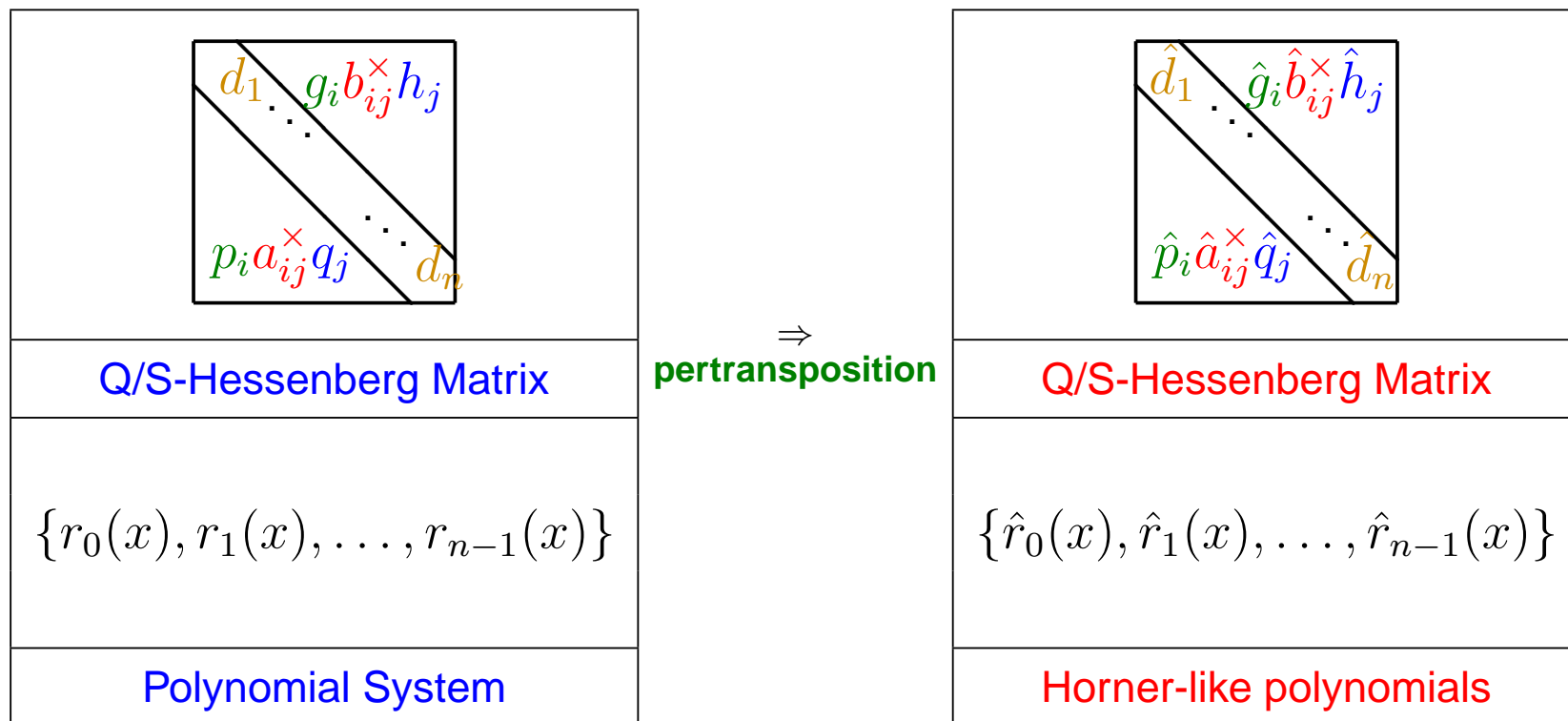
Horner-like Polynomials

- ⇒ A relation between the matrix corresponding to a system of polynomials R and the matrix corresponding to the **Horner-like** polynomials \hat{R} .



Horner-like Polynomials

► **Fact.** A pertransposed quasiseparable matrix is again a quasiseparable matrix.



Generators of a Quasiseparable matrix

► Can be represented in terms of their **generators**:

Diagonal entries

$$d_k \quad k = 1, \dots, n$$

Lower Generators

$$p_k \quad k = 2, \dots, n$$

$$a_k \quad k = 2, \dots, n - 1$$

$$q_k \quad k = 1, \dots, n - 1$$

Upper Generators

$$g_k \quad k = 1, \dots, n - 1$$

$$b_k \quad k = 2, \dots, n - 1$$

$$h_k \quad k = 2, \dots, n$$

► **Example.** In terms of generators, with $n = 5$,

$$\begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ p_2 q_1 & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ p_3 a_2 q_1 & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ p_4 a_3 a_2 q_1 & p_4 a_3 q_2 & p_4 q_3 & d_4 & g_4 h_5 \\ p_5 a_4 a_3 a_2 q_1 & p_5 a_4 a_3 q_2 & p_5 a_4 q_3 & p_5 q_4 & d_5 \end{bmatrix}$$

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