Quasiseparable matrices and polynomials

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Examples of Structured Matrices

Moment Matrices

- **Hankel matrices.** Defined by $\mathcal{O}(n)$ parameters $\{h_k\}$.

$$H = \begin{bmatrix} h_{k+j} \end{bmatrix} = \begin{bmatrix} h_0 & \mathbf{h_1} & h_2 & \cdots & h_{n-1} \\ \mathbf{h_1} & h_2 & & \iddots & \vdots \\ h_2 & & \iddots & & h_{2n-3} \\ \vdots & \iddots & & h_{2n-3} & \mathbf{h_{2n-21}} \\ h_{n-1} & \cdots & h_{2n-3} & \mathbf{h_{2n-2}} & h_{2n-1} \end{bmatrix}$$

- **Toeplitz matrices.** Defined by $\mathcal{O}(n)$ parameters $\{c_k\}$.

$$C = \begin{bmatrix} c_{k-j} \end{bmatrix} = \begin{bmatrix} c_0 & \mathbf{c_{-1}} & \cdots & \cdots & c_{-n+1} \\ c_1 & c_0 & \mathbf{c_{-1}} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & c_0 & \mathbf{c_{-1}} \\ c_{n-1} & \cdots & \cdots & c_1 & c_0 \end{bmatrix}$$

Examples of Structured Matrices

Moment Matrices

$$M = [\langle x^k, x^j \rangle] = \begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle & \dots & \langle 1,x^n\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle 1,x^2\rangle & \dots & \langle x,x^n\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle & \dots & \langle x^2,x^n\rangle \\ \vdots & \vdots & \vdots & & \vdots \\ \langle x^n,1\rangle & \langle x^n,x\rangle & \langle x^n,x^2\rangle & \dots & \langle x^n,x^n\rangle \end{bmatrix}$$

- Real line.

$$\langle p(x), q(x)\rangle = \int_a^b p(x)q(x)w^2(x)dx, \quad \Rightarrow \quad \langle x^k, x^j\rangle = \int_a^b x^{(k+j)}w^2(x)dx,$$

and M is Hankel.

- Unit circle.

$$\langle p(x), q(x)\rangle = \int_{-\pi}^{\pi} p(e^{i\theta}) \cdot \overline{q(e^{i\theta})} w^2(\theta)d\theta \Rightarrow \langle x^k, x^j\rangle = \int_0^{2\pi} x^{(k-j)}w^2(\theta)d\theta,$$

and M is Toeplitz.

Examples of Structured Matrices

Recurrent Matrices

- **Tridiagonal matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$T = \begin{bmatrix} \delta_1 & \gamma_2 & 0 & \cdots & 0 \\ \gamma_2 & \delta_2 & \gamma_3 & \ddots & \vdots \\ 0 & \gamma_3 & \delta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \gamma_n \\ 0 & \cdots & 0 & \gamma_n & \delta_n \end{bmatrix}$$

- **Unitary Hessenberg matrices.** Defined by $\mathcal{O}(n)$ parameters.

$$U = \begin{bmatrix} -\rho_1 {\rho_0}^* & -\rho_2 \mu_1 {\rho_0}^* & \cdots & -\rho_n \mu_{n-1} \ldots \mu_1 {\rho_0}^* \\ \mu_1 & -\rho_2 {\rho_1}^* & \cdots & -\rho_n \mu_{n-1} \ldots \mu_2 {\rho_1}^* \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & -\rho_n \mu_{n-1} {\rho_{n-2}}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n {\rho_{n-1}}^* \end{bmatrix}$$

Orthogonal polynomials related to structured matrices

Moment matrix	Recurrent matrix	Polynomials $r_k(x)$
Hankel matrices	tridiagonal matrices	Real–orthogonal polynomials
Toeplitz matrices	unitary Hessenberg matrices	Szegö polynomials

Orthogonal polynomials related to structured matrices

Moment matrix	Recurrent matrix	Polynomials $r_k(x)$
Hankel matrices	tridiagonal matrices	Real–orthogonal polynomials
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- Related to tridiagonal matrices via $\mathbf{r_k}(x) = \det(xI - A)_{(\mathbf{k}\times\mathbf{k})}$ where

$$A = \begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & & \ddots & \frac{\delta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} \end{bmatrix}$$

- **Three–term recurrence relations**

$$\mathbf{r_k}(x) = (\alpha_k \mathbf{x} - \delta_k)\mathbf{r_{k-1}}(x) - \gamma_k \mathbf{r_{k-2}}(x)$$

Orthogonal polynomials related to structured matrices

Moment matrix	Recurrent matrix	Polynomials $r_k(x)$
Hankel matrices	tridiagonal matrices	Real–orthogonal polynomials
Toeplitz matrices	unitary Hessenberg matrices	Szegö polynomials

➠ Related to unitary Hessenberg matrices via $\mathbf{r_k}(x) = \det(xI - A)_{(\mathbf{k}\times\mathbf{k})}$ where

$$A = \begin{bmatrix} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & -\rho_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & -\rho_3\mu_2\rho_1^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ 0 & \mu_2 & -\rho_3\rho_2^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \\ \vdots & \ddots & \mu_3 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{n-1}\rho_{n-2}^* & -\rho_n\mu_{n-1}\rho_{n-2}^* \\ 0 & \cdots & \cdots & 0 & \mu_{n-1} & -\rho_n\rho_{n-1}^* \end{bmatrix}$$

Orthogonal polynomials related to structured matrices

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Two–term recurrence relations

$$\begin{bmatrix} G_{k+1}(x) \\ \mathbf{r_{k+1}}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}{}^{*} \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{x} \end{bmatrix} \begin{bmatrix} G_k(x) \\ \mathbf{r_k}(x) \end{bmatrix}.$$

Three–term recurrence relations

$$\mathbf{r_k}(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) \mathbf{r_{k-1}}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \right) \mathbf{r_{k-2}}(x)$$

Generalizations of these structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	????????????????????

Generalizations of these structures

Matrix class	Generalized class
Hankel matrices Toeplitz matrices	matrices with displacement structure
tridiagonal matrices unitary Hessenberg matrices	quasiseparable matrices

Previous work in fast algorithms extending those for Vandermonde matrices

polynomial–Vandermonde matrix	Björck–Pereyra–type linear system solver	Traub–type inversion algorithm
Vandermonde	Björck-Pereyra(1970)	Traub (1966)
Chebyshev-Vandermonde	Reichel-Opfer (1991)	Gohberg-Olshevsky (1994)
three-term Vandermonde	Higham (1988,90)	Calvetti-Reichel (1993)
Szegö-Vandermonde	BEGKO (2006)	Olshevsky (2001)

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Szegö-Vandermonde	BEGKO (2006)	Olshevsky (2001)
Quasiseparable-Vandermonde	BEGKO (2007)	BEGOT (2007)
		BEGOZ (2007)

Questions

- **Polynomials.**
  - Are there general 3–term recurrence relations that contain the 3–term recurrence relations satisfied by real–orthogonal polynomials and Szegö polynomials as special cases?
  - Are there (Szegö–type) 2–term recurrence relations, i.e. of the form

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1} \\ (\delta_k x + \theta_k) r_{k-1} \end{bmatrix}$$

    for real–orthogonal polynomials?

- **Matrices.**
  - Describe classes of matrices whose characteristic polynomials satisfy **general 3–term recurrence relations**.
  - Describe classes of matrices whose characteristic polynomials satisfy **general 2–term recurrence relations**.

Quasiseparable Matrices

- **Definition.** A matrix C is $(H,1)$**–quasiseparable** if it is upper Hessenberg and

$$\max \text{Rank} C_{12} = 1$$

where the maxima are taken over all symmetric partitions of the form

$$C = \left[\begin{array}{c|c} * & C_{12} \\ \hline * & * \end{array}\right]$$

- **Previous Work.** Gohberg-Kaashoek-Lerer, Dewilde, Gohberg-Eidelman, Van Barel et al, Tyrtyshnikov et al, Bini et al, Gu-Chandrasekaran et al.

Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k\times k})$ associated with C is the **monomials** with recurrence relations

$$r_k(x) = x r_{k-1}(x)$$

- The matrix C is $(H, 1)$**–quasiseparable**.

Important Special Cases

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Important Special Cases

$$C = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k\times k})$ associated with C are **real orthogonal polynomials** with recurrence relations

$$r_k(x) = \frac{1}{q_k}(x - d_k) r_{k-1}(x) - \frac{g_{k-1}}{q_k} r_{k-2}(x)$$

- The matrix C is $(H, 1)$**–quasiseparable**.

Important Special Cases

Tridiagonal

$$C = \begin{bmatrix} d_1 & g_1 & 0 & 0 & 0 \\ q_1 & d_2 & g_2 & 0 & 0 \\ 0 & q_2 & d_3 & g_3 & 0 \\ 0 & 0 & q_3 & d_4 & g_4 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

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Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

- The system of polynomials $r_k(x) = \det(xI - C_{k\times k})$ associated with C are the **Szegö polynomials** with recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ x r_{k-1}(x) \end{bmatrix}$$

- The matrix C is $(H,1)$**–quasiseparable**.

Important Special Cases

Unitary Hessenberg

$$C = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

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Important Special Cases

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A generator representation

- The following matrix is $(H,1)$**–quasiseparable**:

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & p_4q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & p_5q_4 & d_5 \end{bmatrix}$$

- This **generator representation** exists for any $(H,1)$**–quasiseparable** matrix.

$(H,1)$–quasiseparable $\Leftrightarrow$ Generator representation

- Algorithms operating on generators allow use of **low storage**.

Questions

- **Polynomials.**
  - Are there general 3–term recurrence relations that contain the 3–term recurrence relations satisfied by real–orthogonal polynomials and Szegö polynomials as special cases?
  - Are there (Szegö–type) 2–term recurrence relations, i.e. of the form

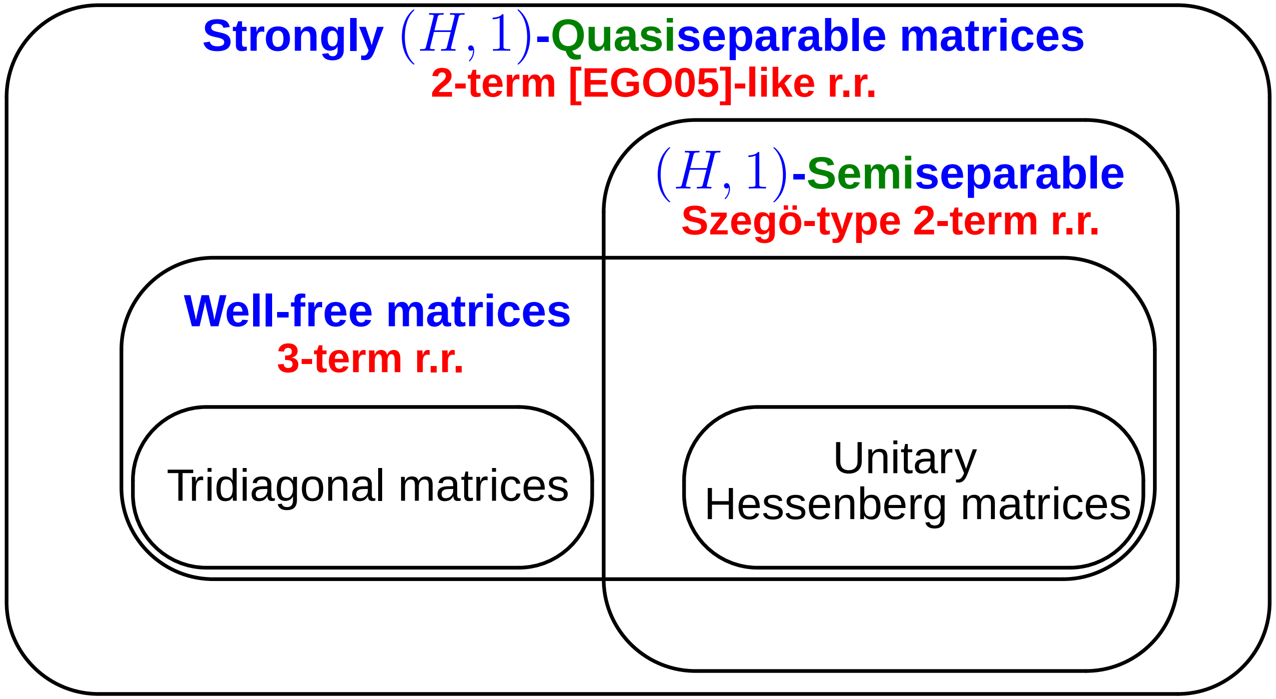
$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1} \\ (\delta_k x + \theta_k) r_{k-1} \end{bmatrix}$$

    for real–orthogonal polynomials?

- **Matrices.**
  - Describe classes of matrices whose characteristic polynomials satisfy **general 3–term recurrence relations**.
  - Describe classes of matrices whose characteristic polynomials satisfy **general 2–term recurrence relations**.

Subclasses of $(H,1)$–quasiseparable matrices

Corresponding recurrence relations



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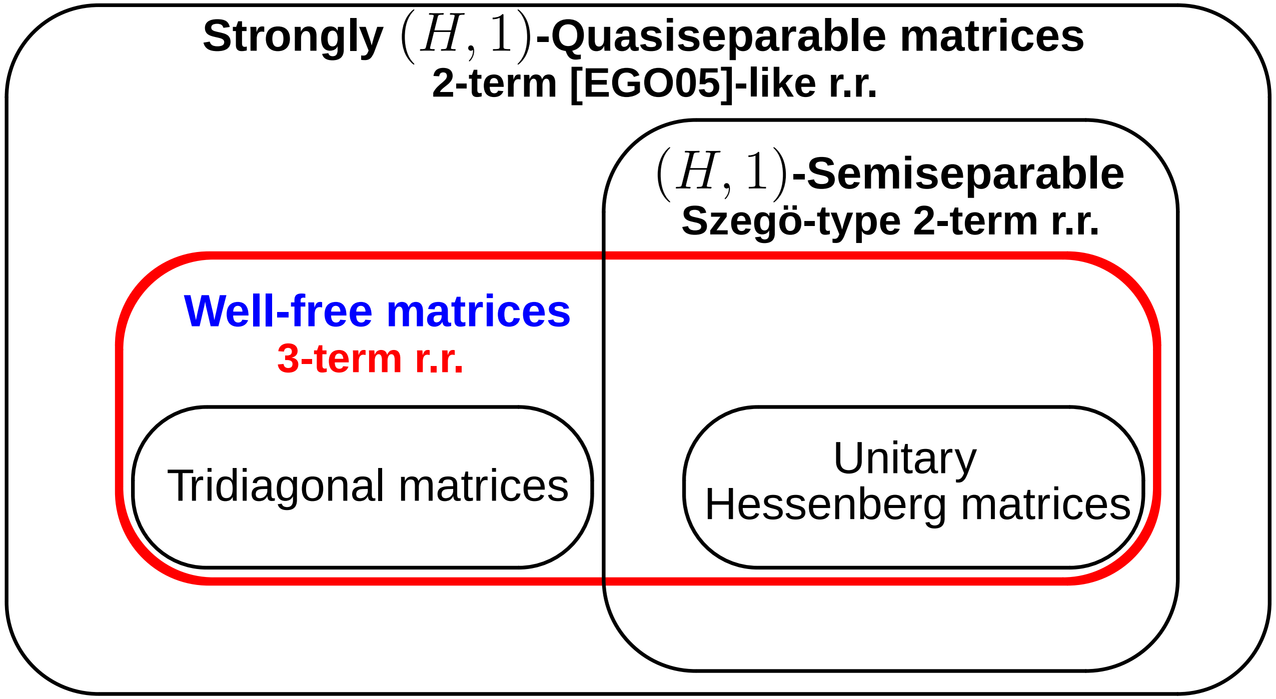


Three-term recurrence relations.

We begin by considering the class of polynomials satisfying

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\beta_k x + \gamma_k) r_{k-2}(x)$$

They generalize:

- Real-orthogonal polynomials: $\beta_k = 0$

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k r_{k-2}(x)$$

- Szegö polynomials (orthogonal on the unit circle): $\gamma_k = 0$

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}\right) r_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x\right) r_{k-2}(x)$$

Three-term recurrence relations.

We begin by considering the class of polynomials satisfying

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\boxed{\beta_k} x + \boxed{\gamma_k}) r_{k-2}(x) \tag{1}$$

They generalize:

- Real-orthogonal polynomials: $\boxed{\beta_k} = 0$

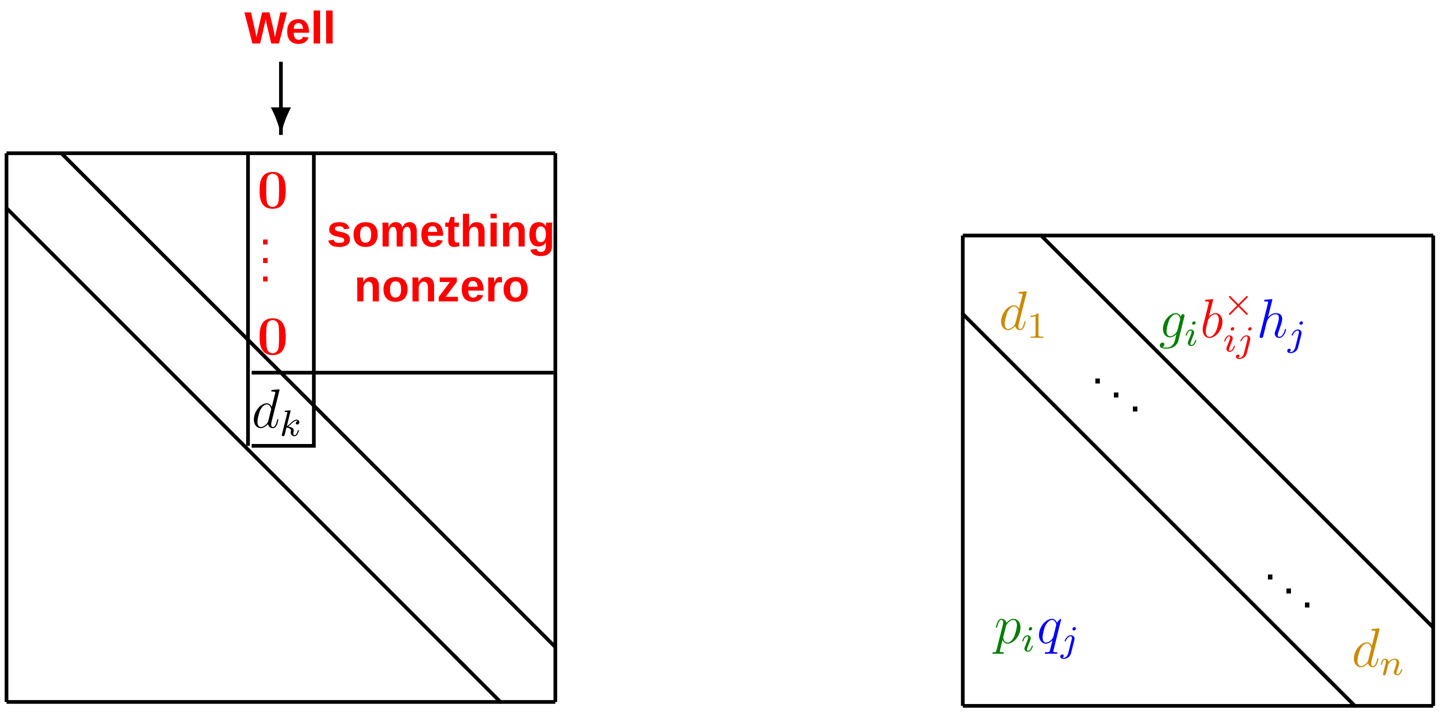
$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \boxed{\gamma_k} r_{k-2}(x)$$

- Szegö polynomials (orthogonal on the unit circle): $\boxed{\gamma_k} = 0$

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k} \right) r_{k-1}(x) - \left(\boxed{\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k}} \cdot x \right) r_{k-2}(x)$$

A problem: full characterization of the matrix class corresponding to (1).

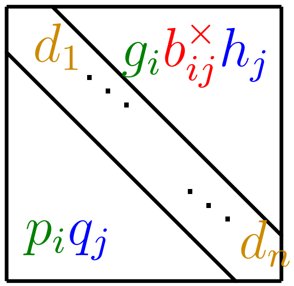
First matrix class. Well-free matrices



Well-free $\Leftrightarrow h_j \neq 0$

- Irreducible tridiagonal are well-free.
- Unitary Hessenberg are well-free.

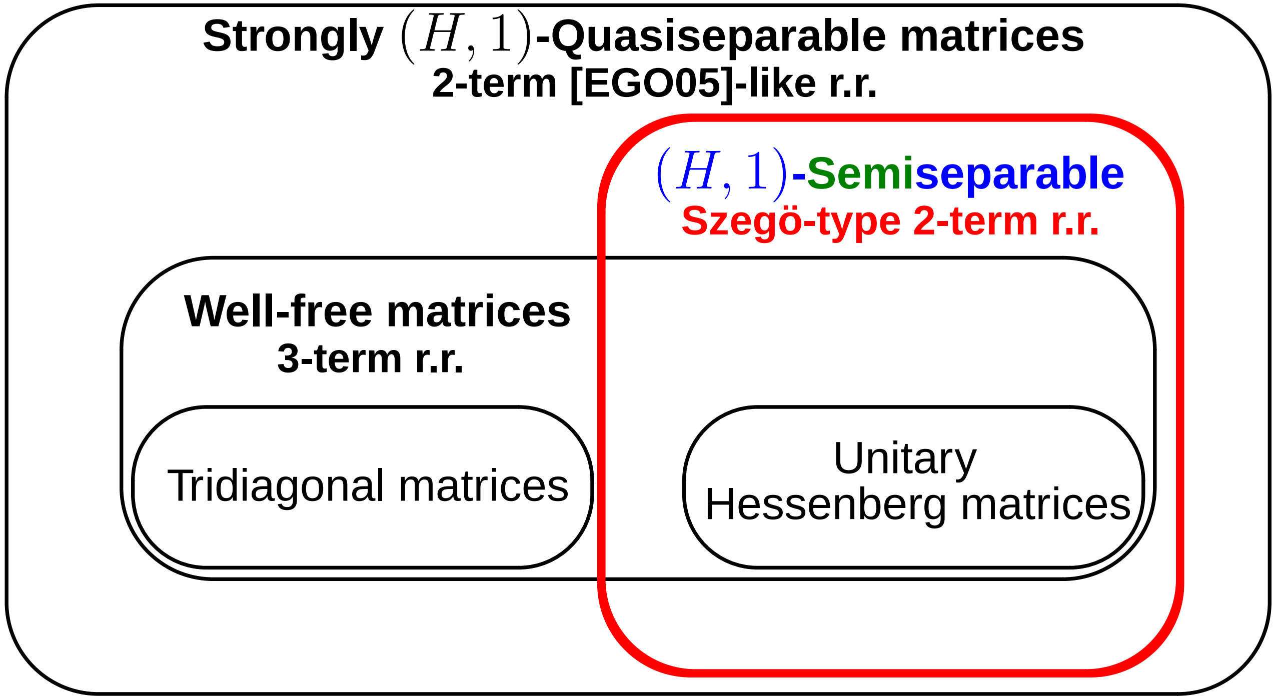
Well-free matrices & 3-term recurrence relations. Equivalence

		$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\beta_k x + \gamma_k) r_{k-2}(x)$
Well-free $(H,1)$–quasiseparable matrix	$\Leftrightarrow$ **conversions**	3-term recurrence relations
$\{p_k, q_k, d_k, g_k, b_k, h_k\}$ $\mathbf{h_k \neq 0}$		$\{\alpha_k, \beta_k, \gamma_k, \delta_k\}$
Quasiseparable generators		Recurrence relation coefficients

Restrictions: $h_k \neq 0$

Subclasses of $(H,1)$–quasiseparable matrices

Corresponding recurrence relations



Second matrix class. Semiseparable matrices

- R is called *order* (r_L, r_U)-***semiseparable*** if for some **small** r_L, r_U we have

$$R = D + tril(R_L) + triu(R_U),$$

where rank$R_L = r_L$, rank$R_U = r_U$, with some R_L, R_U.

- Example of order $(1,1)$-**semiseparable**:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Second matrix class. Semiseparable matrices

- R is called *order* (r_L, r_U)-***semiseparable*** if for some **small** r_L, r_U we have

$$R = D + tril(R_L) + triu(R_U),$$

where $\text{rank}R_L = r_L, \quad \text{rank}R_U = r_U, \quad$ with some R_L, R_U.

- Example of order $(1,1)$-**semiseparable**:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Second matrix class. Semiseparable matrices

- R is called *order* (r_L, r_U)-***semiseparable*** if for some **small** r_L, r_U we have

$$R = D + tril(R_L) + triu(R_U),$$

where $\text{rank} R_L = r_L, \quad \text{rank} R_U = r_U, \quad$ with some R_L, R_U.

- Example of order $(1,1)$-**semiseparable**:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Second matrix class. Semiseparable matrices

- R is called *order* (r_L, r_U)-***semiseparable*** if for some **small** r_L, r_U we have

$$R = D + tril(R_L) + triu(R_U),$$

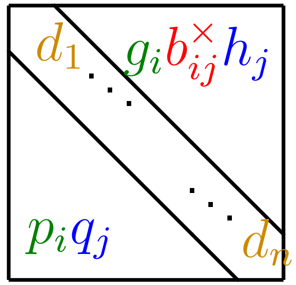
where rank$R_L = r_L$, rank$R_U = r_U$, with some R_L, R_U.

- Example of order $(1,1)$-**semiseparable**:

$$R_L = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}, R_U = \begin{bmatrix} c_1d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ c_2d_1 & c_2d_2 & c_2d_3 & c_2d_4 \\ c_3d_1 & c_3d_2 & c_3d_3 & c_3d_4 \\ c_4d_1 & c_4d_2 & c_4d_3 & c_4d_4 \end{bmatrix}$$

$$R = \begin{bmatrix} d_1 & c_1d_2 & c_1d_3 & c_1d_4 \\ a_2b_1 & d_2 & c_2d_3 & c_2d_4 \\ a_3b_1 & a_3b_2 & d_3 & c_3d_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & d_4 \end{bmatrix}$$

Semiseparable matrices & Szegö-type 2-term recurrence relations. Equivalence



$(H,1)$–quasiseparable matrix

$\{p_k, q_k, d_k, g_k, b_k, h_k\}$

$\mathbf{b_k \neq 0}$

Quasiseparable generators

$\Leftrightarrow$
conversions

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1} \\ (\delta_k x + \theta_k) r_{k-1} \end{bmatrix}$$

Szegö-type recurrence relations

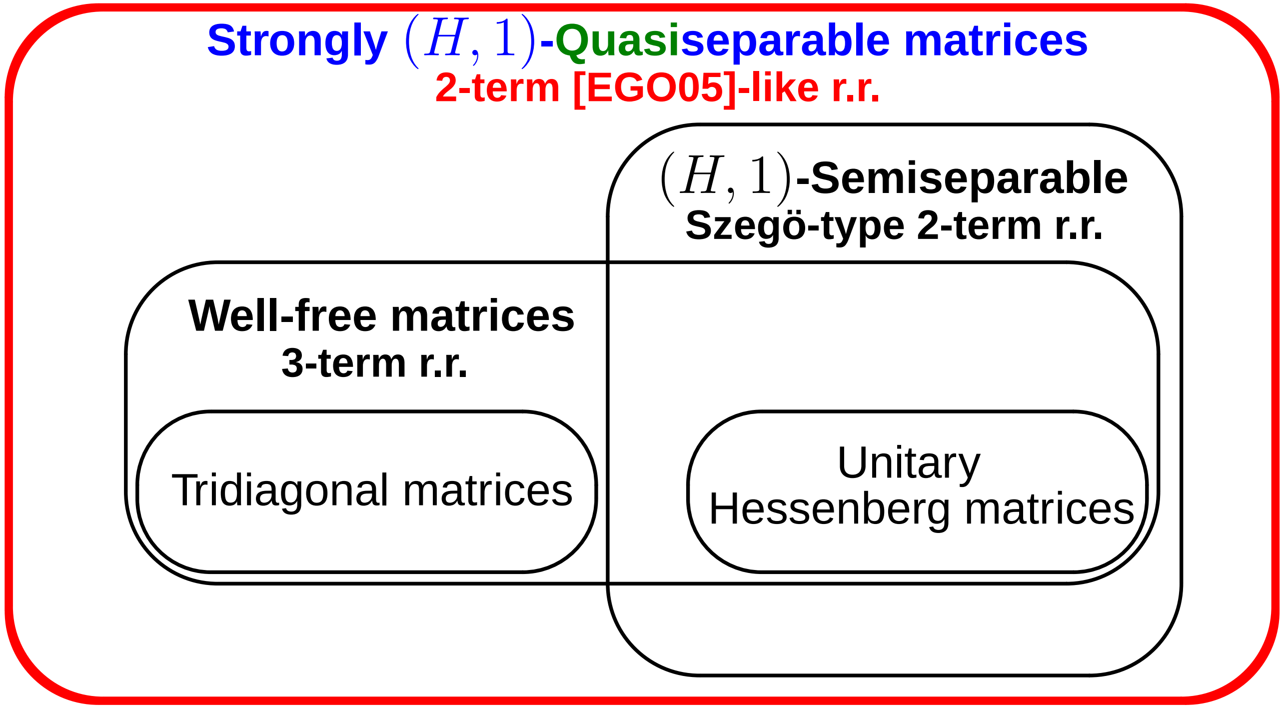
$\{\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k\}$

Recurrence relation coefficients

Restrictions: $b_k \neq 0$

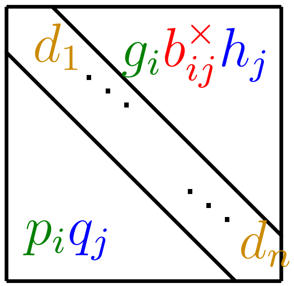
Full Characterization of $(H,1)$–quasiseparable matrices

Corresponding recurrence relations



Third matrix class. Quasiseparable matrices & [EGO05]-type 2-term recurrence relations. Equivalemce.

A complete characterization of Hessenberg order-one quasiseparable matrices



$(H, 1)$–quasiseparable matrix

$$\{p_k, q_k, d_k, g_k, b_k, h_k\}$$

Quasiseparable generators

⇔ **conversions**

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}$$

[EGO05]-type recurrence relations

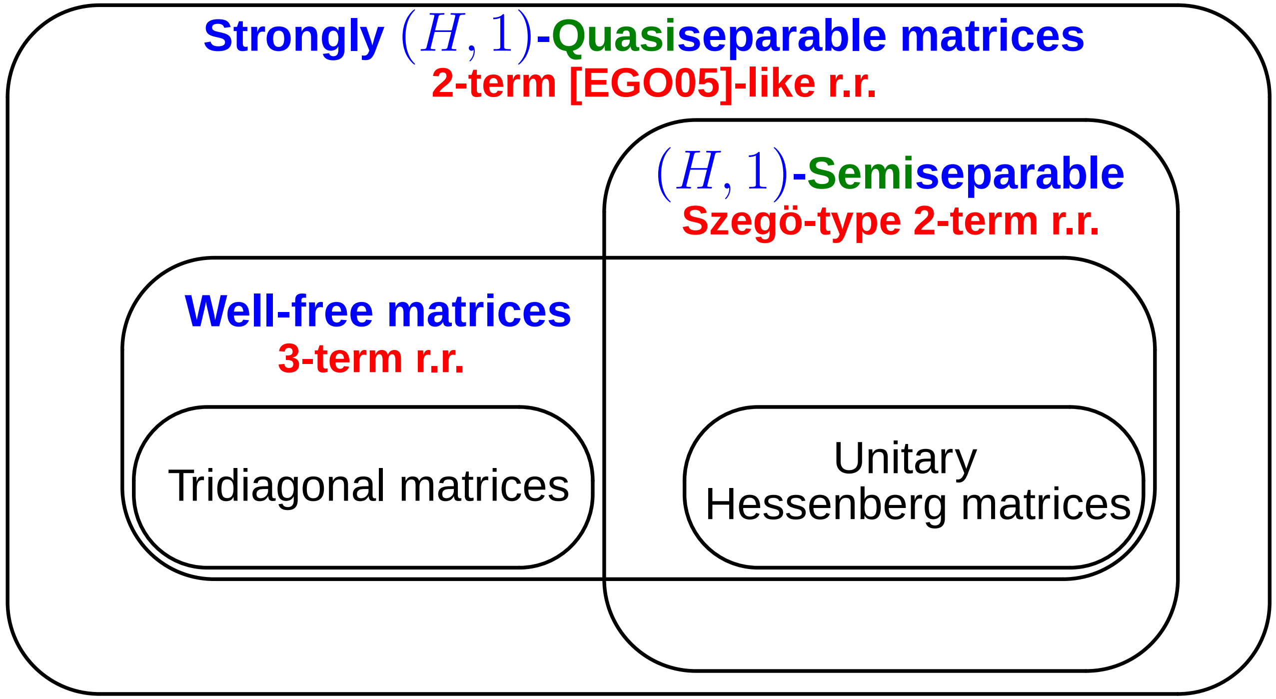
$$\{\alpha_k, \beta_k, \gamma_k, \delta_k, \theta_k\}$$

Recurrence relation coefficients

Restrictions: NONE.

Subclasses of $(H,1)$–quasiseparable matrices

Corresponding recurrence relations

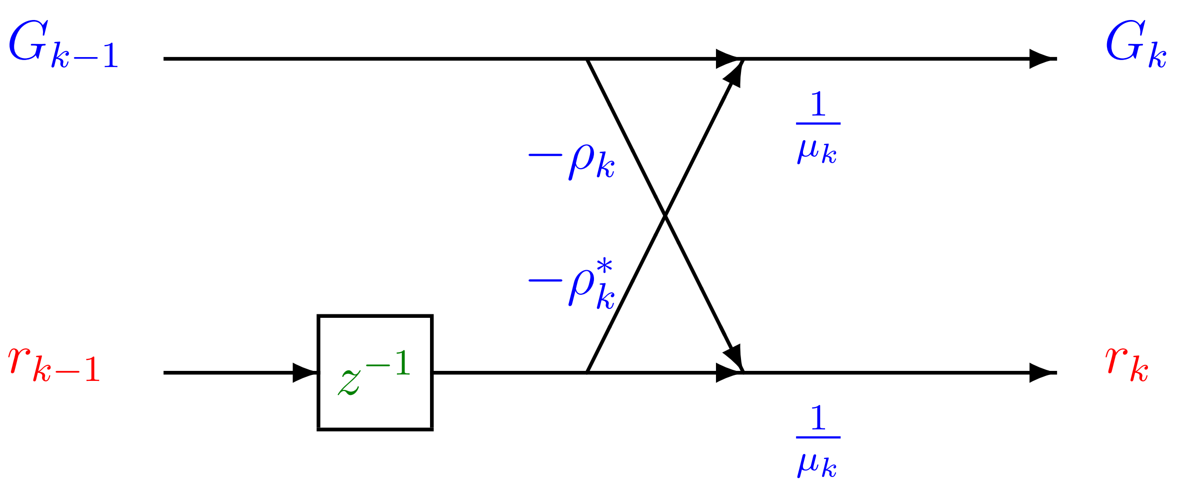


Signal flow graphs

Szegö polynomials satisfy well–known two–term recurrence relations,

$$\begin{bmatrix} G_k(z^{-1}) \\ r_k(z^{-1}) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(z^{-1}) \\ \mathbf{z^{-1}} r_{k-1}(z^{-1}) \end{bmatrix}.$$

can be realized as

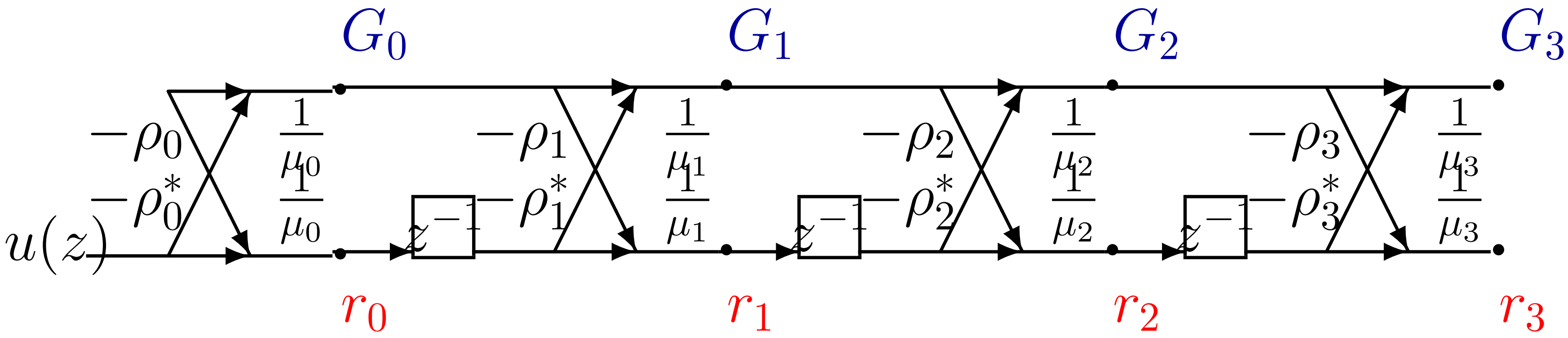


Well-known lattice filter characterization of Unitary Hesenberg

- THEOREM: Matrix A is (almost) unitary Hessenberg if and only if polynomials

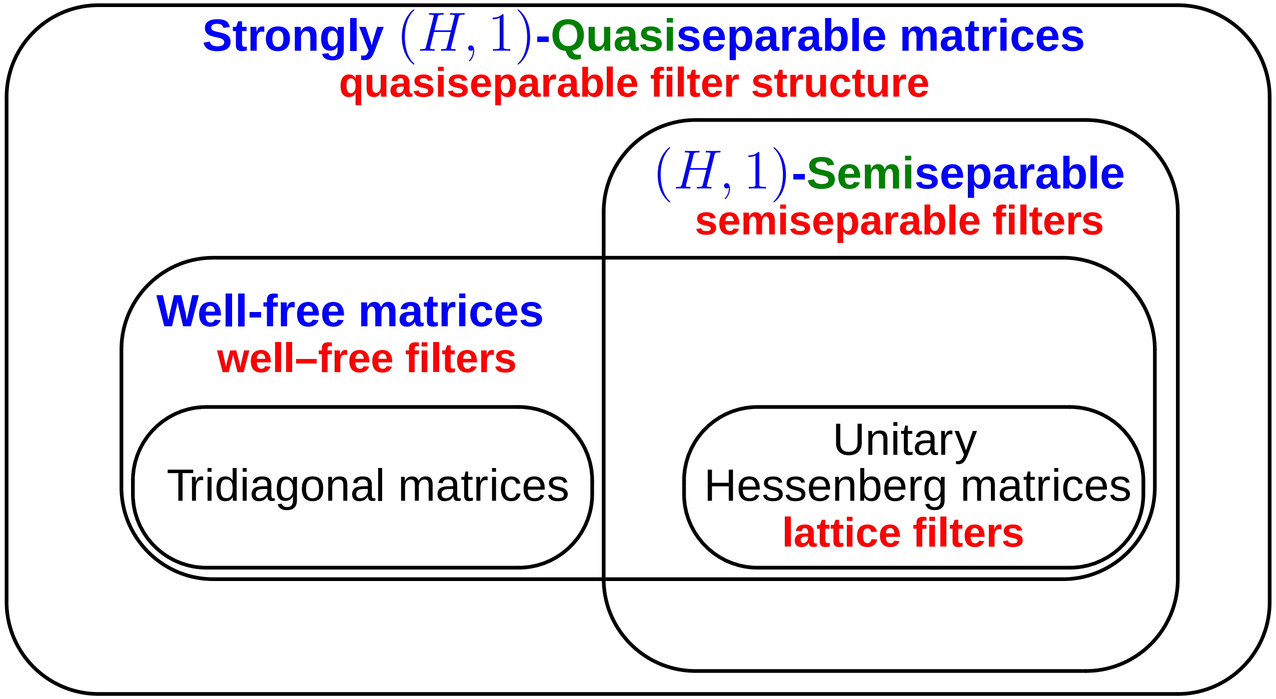
$$r_k(x) = \det(xI - A)_{(k\times k)}$$

admit the following lattice realization (with some conditions).



Subclasses of $(H,1)$–quasiseparable matrices

Corresponding digital filter structures

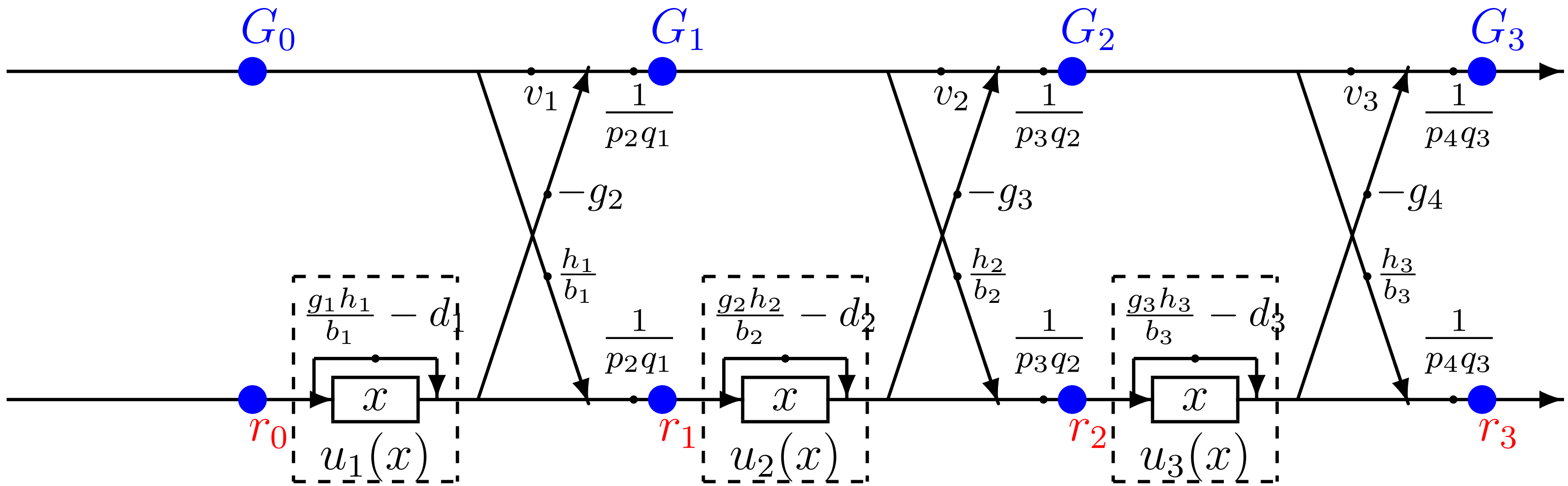


Semiseparable filter structures

➠ THEOREM: Matrix A is $(H, 1)$–semiseparable if and only if polynomials

$$r_k(x) = \det(xI - A)_{(k \times k)}$$

admit the following lattice-like realization

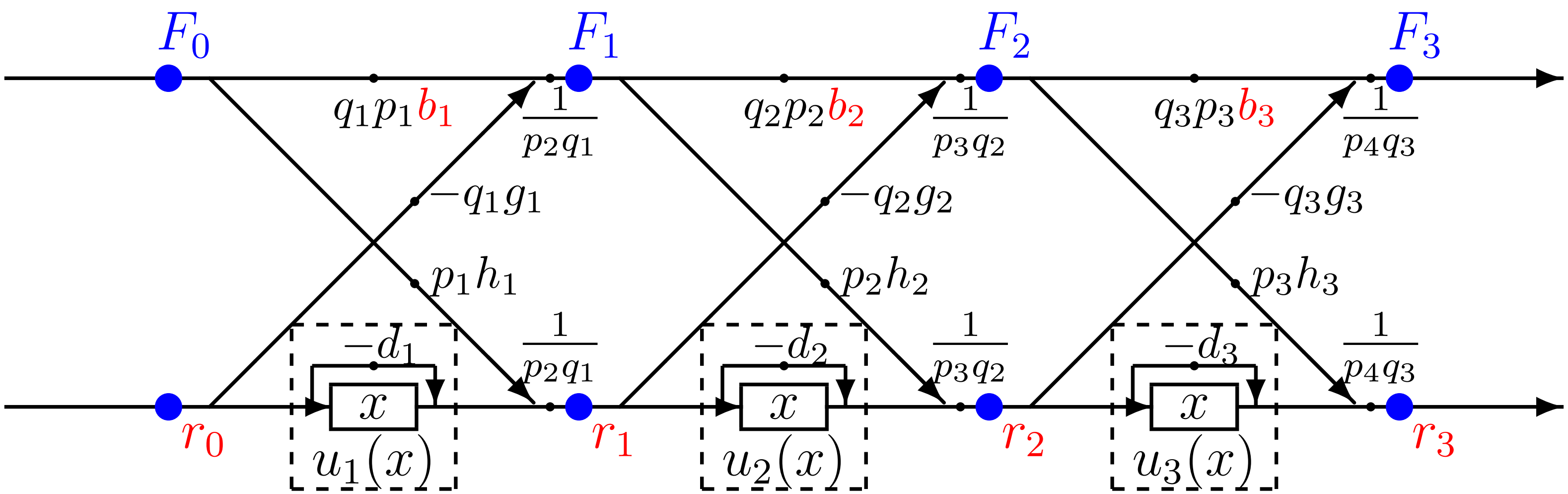


Quasiseparable filter structures

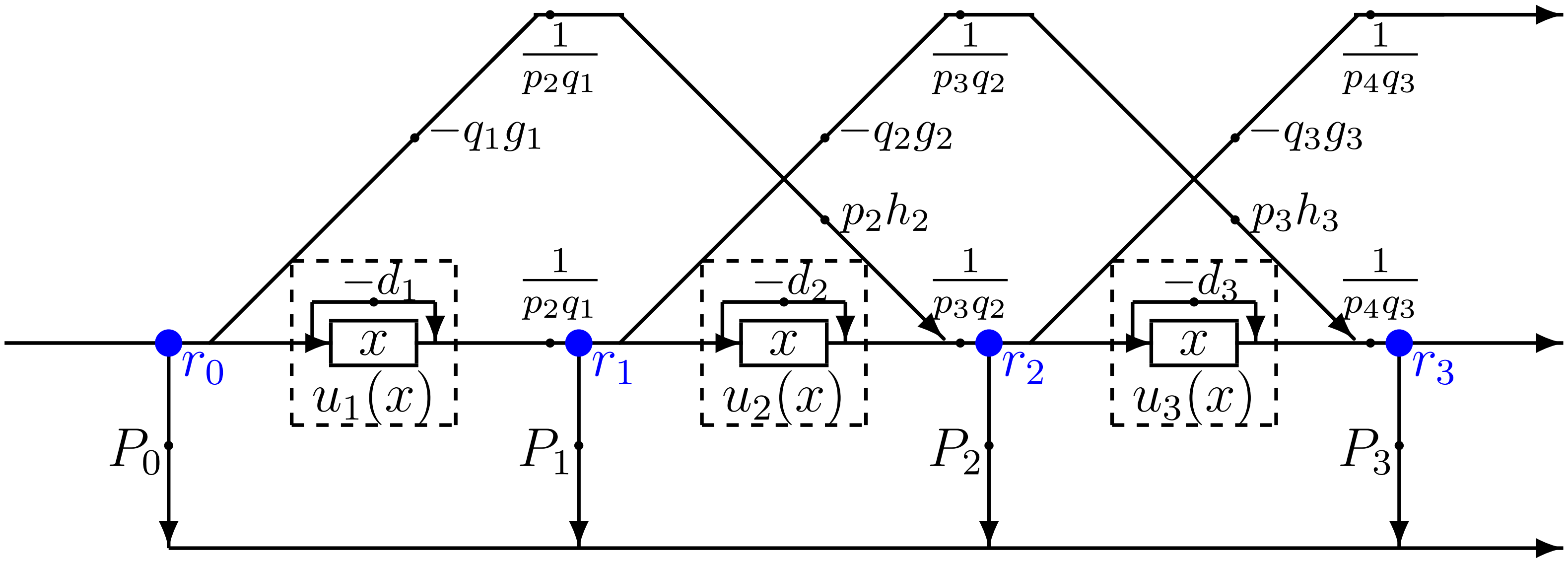
➠ THEOREM: Matrix A is $(H,1)$–quasiseparable if and only if polynomials

$$r_k(x) = \det(xI - A)_{(k\times k)}$$

admit the following lattice-like realization



Signal flow graph for real OP using quasiseparable filter structure



$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & p_4q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & p_5q_4 & d_5 \end{bmatrix}, \qquad \boxed{\mathbf{b_k = 0}}$$

Quasiseparable matrices and polynomials

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Joint work with:
Yuli Eidelman, Israel Gohberg, & Vadim Olshevsky.

Hadamard–Sylvester and Pseudo–noise matrices are equivalent

Joint work with Vadim Olshevsky & Lev Sakhnovich

- A **Hadamard matrix** is one whose entries are ± 1 and satisfy $H_n^T H_n = nI_n$.

  A **Hadamard–Sylvester matrix** is a Hadamard matrix built from the recursion $H_1 = [1]$,

  $$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}.$$

- The output of a linear–shift register corresponding to a primitive polynomial is called a **Pseudo–noise sequence**.

  A **Pseudo–noise matrix** is a padded, circulant Hankel matrix whose rows are Pseudo–noise sequences.

- **Theorem.** Hadamard–Sylvester matrices and Pseudo–noise matrices are **equivalent**; i.e. one can be obtained from the other via row and column permutations.

Structure–preserving perturbations of matrices self–adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

- For a Hermitian, invertible (not necessarily positive definite) matrix H, one defines the **indefinite inner product** via

$$[x, y]_H = (Hx, y) = y^* Hx$$

where $(\cdot, \cdot)$ denotes the Euclidean inner product.

- One defines **self–adjoint with respect to an indefinite inner product** in an analogous way to the definition for classical self-adjoint.

Euclidean Inner Product	Indefinite Inner Product
$(Ax, y) = (x, Ay)$	$[Ax, y]_H = [x, Ay]_H$
$A = A^*$	$HA = A^*H$

Structure–preserving perturbations of matrices self–adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

- Pairs of matrices (A, H) have a **canonical form** (J, P) [Gohberg–Lancaster–Rodman 1983], with

$$J = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus \widetilde{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \widetilde{J}(\lambda_\beta)$$

and

$$P = \epsilon_1 P_1 \oplus \cdots \oplus \epsilon_\alpha P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P(\lambda_\beta)$$

where $\lambda_1, \ldots, \lambda_\alpha \in \mathbb{R}$, $J(\lambda)$ is a Jordan block, and $\widetilde{J}(\lambda) = J(\lambda) \oplus J(\bar{\lambda})$.

- The matrix that reduces to this canonical form, T such that

$$T^{-1}AT = J, \qquad T^*HT = P$$

has columns that are not only a Jordan basis of A, they also bring H into P, and we call such a basis a **canonical Jordan basis of (A, H)**.

Structure–preserving perturbations of matrices self–adjoint with respect to an indefinite inner products

Joint work with Vadim Olshevsky & Upendra Prasad

➠ **Theorem.** Let $A_0 \in \mathbb{C}^{n\times n}$ be a fixed H_0-selfadjoint matrix. Let

$$\{\{f_r^{(k,s)}\}_{r=0}^{m_k(A_0,\lambda_s)-1}\}_{s=1,k=1}^{s=\beta,k=\dim\ker(A_0-\lambda_s I)}$$

be a fixed canonical Jordan basis of A_0. There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any H-selfadjoint matrix A such that A has the same Jordan structure as A_0 and

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists a canonical Jordan basis

$$\{\{g_r^{(k,s)}\}_{r=0}^{m_k(A,\lambda_s)-1}\}_{s=1,k=1}^{s=\beta,k=\dim\ker(A-\lambda_s I)}$$

of A such that

$$\|g_r^{(k,s)} - f_r^{(k,s)}\| \leqslant K\left(\|A - A_0\| + \|H - H_0\|\right)$$

for all k, s, r within their ranges.

Quasiseparable matrices and polynomials

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Joint work with:
Yuli Eidelman, Israel Gohberg, & Vadim Olshevsky.

Supplemental Slides

Introduction

Vandermonde matrices and algorithms

- **Definition.** For a set of nodes, a **Vandermonde matrix** is defined by

$$x = \{x_1, x_2, \ldots, x_n\}$$
$$\Downarrow$$
$$V(x) = \begin{bmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{n-1} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{n-1} \\ 1 & x_3 & {x_3}^2 & \cdots & {x_3}^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{n-1} \end{bmatrix}$$

- **Note.** The n^2 entries of $V(x)$ are defined by only n parameters. This allows the design of **fast algorithms**.

Fast Algorithms for Vandermonde matrices

- **Björck-Pereyra algorithm (1970).** The solution a of the linear system $V(x)a = f$ is computed by Björck-Pereyra as the successive product of sparse matrices with a vector, as

$$a = V(x)^{-1} f = U_1^{-1} \dots U_{n-1}^{-1} L_{n-1}^{-1} \dots L_1^{-1} f.$$

- **The Traub algorithm (1966).** The inverse of a Vandermonde matrix can be computed from the formula

$$V(x)^{-1} = \widetilde{I} V_R(x)^T D.$$

where the elements of $V_R(x)^T$ are defined by the **Horner polynomials**, and can be computed quickly using the **two-term recurrence relations**

$$r_0(x) = P_n, \qquad r_k(x) = x r_{k-1}(x) + P_{n-k}$$

- Both of these algorithms are **fast** ($\mathcal{O}(n^2)$ operations vs. $\mathcal{O}(n^3)$ required by Gaussian elimination) and **accurate**.

Generalizations of these algorithms

Polynomial–Vandermonde matrices

➠ **Definition.** For sets of polynomials and nodes, define a **polynomial–Vandermonde matrix**:

$$x = \{x_1, x_2, \ldots, x_n\}$$

$$R = \{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$$

$$\Downarrow$$

$$V_R(x) = \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}$$

➠ **Note.** If the polynomial system R satisfies nice recurrence relations, then the n^2 entries of $V_R(x)$ can be defined by only $\mathcal{O}(n)$ parameters.

Fast algorithms for polynomial–Vandermonde matrices

Previous work

polynomial–Vandermonde matrix	Björck-Pereyra-type	Traub-type
Vandermonde	Björck-Pereyra(1970)	Traub (1966)
Chebyshev-Vandermonde	Reichel-Opfer (1991)	Gohberg-Olshevsky (1994)
three-term Vandermonde	Higham (1988,90)	Calvetti-Reichel (1993)
Szegö-Vandermonde	BEGKO (2006)	Olshevsky (2001)

Extensions of these algorithms

- A generalization of the Björck-Pereyra algorithm that includes all of these previous cases.
- A generalization of the Traub algorithm that includes all of these previous cases.

How?

Fast algorithms for polynomial–Vandermonde matrices

Previous work

polynomial–Vandermonde matrix	Björck-Pereyra-type	Traub-type
Vandermonde	Björck-Pereyra(1970)	Traub (1966)
Chebyshev-Vandermonde	Reichel-Opfer (1991)	Gohberg-Olshevsky (1994)
three-term Vandermonde	Higham (1988,90)	Calvetti-Reichel (1993)
Szegö-Vandermonde	BEGKO (2006)	Olshevsky (2001)
Quasiseparable-Vandermonde	BEGKO (2007)	BEGOT (2007)
		BEGOZ (2007)

Extensions of these algorithms

- A generalization of the Björck-Pereyra algorithm that includes all of these previous cases.
- A generalization of the Traub algorithm that includes all of these previous cases.

QUASISEPARABLE MATRICES

Quasiseparable-Vandermonde matrices

- **Definition.** A **Quasiseparable-Vandermonde matrix** is of the form

$$V_R = \begin{bmatrix} r_0(x_1) & r_1(x_1) & r_2(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & r_2(x_2) & \cdots & r_{n-1}(x_2) \\ r_0(x_3) & r_1(x_3) & r_2(x_3) & \cdots & r_{n-1}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_0(x_n) & r_1(x_n) & r_2(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}$$

where the polynomials $r_k(x)$ defined by $r_k(x) = \det(xI - C_{k\times k})$ correspond to an $(H,1)$**–quasiseparable** matrix C.

- The class of $(H,1)$**–quasiseparable** polynomials contains as subclasses the classes of **real–orthogonal polynomials** and **Szegö polynomials**.

A Björck–Pereyra–like algorithm for quasiseparable–Vandermonde matrices

Joint work with Yuli Eidelman, Israel Gohberg, Israel Koltracht, & Vadim Olshevsky

Björck–Pereyra–like algorithm for quasiseparable–Vandermonde matrices

- Like the Björck–Pereyra algorithm, the generalization is based on the formula

$$V_R^{-1} = U_1^{-1} \dots U_{n-1}^{-1} \, L_{n-1}^{-1} \dots L_1^{-1} \, ,$$

with

$$U_k^{-1} = \text{diag}\{ I_{k-1}, \left[\begin{array}{c|ccc} \frac{1}{\alpha_0} & & & \\ 0 & & C - x_k I & \\ \vdots & & & \\ 0 & & & \\ \hline 0 & 0 & \cdots \; 0 & \frac{1}{\alpha_{n-k}} \end{array} \right] \}$$

$$L_k^{-1} = \left[\begin{array}{c|cccc} I_{k-1} & & & & \\ \hline & 1 & & & \\ & & \frac{1}{x_{k+1} - x_k} & & \\ & & & \ddots & \\ & & & & \frac{1}{x_n - x_k} \end{array} \right] \left[\begin{array}{c|cccc} I_{k-1} & & & & \\ \hline & 1 & & & \\ & -1 & 1 & & \\ & \vdots & & \ddots & \\ & -1 & & & 1 \end{array} \right]$$

Complexity of the Björck–Pereyra–like algorithm

- To design a fast algorithm, we need **fast multiplication of C by a vector**.

  The **classical Björck–Pereyra algorithm** (monomial case), C is **bidiagonal**.

  In the **Szegö** case, C is **unitary Hessenberg**, and hence admits a convenient **Schur factorization**.

- How does one multiply a $(H, 1)$**–quasiseparable** matrix by a vector in $\mathcal{O}(n)$ operations?

Complexity of the Björck–Pereyra–like algorithm

Proposition. An $(H,1)$–quasiseparable matrix C admits the decomposition

$$C = L + U$$

for a lower–bidiagonal matrix L and

$$U = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & g_{n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & \widetilde{B}^{-1} & \\ 0 & & & \\ \hline 0 & 0 & \cdots & 0 \end{array}\right] \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h_n \end{bmatrix}$$

and

$$\widetilde{B} = \begin{bmatrix} I & -b_2 & & \\ & \ddots & \ddots & \\ & & I & -b_{n-1} \\ & & & I \end{bmatrix}$$

Complexity of the Björck–Pereyra–like algorithm

- Thus, in the **quasiseparable** case, C can be multiplied by a vector at the cost of a multiplication by a bidiagonal matrix, two diagonal scalings, and a back–substitution with a bidiagonal matrix.
- This leads to an $\mathcal{O}(n)$ algorithm.
- This implementation coincides with the algorithm derived differently by **Eidelman and Gohberg** (1999).
- Thus the cost of the Björck–Pereyra–like algorithm is $\mathcal{O}(n^2)$ arithmetic operations.

Numerical Illustrations - Björck-Pereyra

ADD HERE, IF TIME.

- We compare the **forward error** of the solutions $\hat{x}$ from MATLAB in double precision via

$$e = \frac{\|x - \hat{x}\|_2}{\|x\|_2},$$

  with x, the "exact" solution using MATLAB's `vpa()` command for software-implemented arbitrary digit arithmetic.

- **GE** - Gaussian elimination via MATLAB's backslash command.
- **BP-QS** - Björck–Pereyra–like algorithm.
- **BP-QS-L** - Björck–Pereyra–like algorithm with nodes ordered via the **Leja ordering**. (Reichel, Higham)

Confederate Matrices

- **Definition** For polynomials $R = \{r_0(x), r_1(x), \ldots, r_n(x)\}$ satisfying n-term recurrence relations

$$r_k(x) = \alpha_k \cdot x r_{k-1}(x) - a_{k-1,k} \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \cdots - a_{0,k} \cdot r_0(x).$$

and the polynomial

$$b(x) = b_0 \cdot r_0(x) + b_1 \cdot r_1(x) + \cdots + b_{n-1} \cdot r_{n-1}(x) + b_n \cdot r_n(x)$$

define the **confederate matrix** of b with respect to R by

$$C_R(b) = \begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n} \end{bmatrix}$$

Confederate Matrices

$$
\begin{aligned}
r_1(x) &= \alpha_1 \cdot x r_0(x) - a_{0,1} \cdot r_0(x) \\
r_2(x) &= \alpha_2 \cdot x r_1(x) - a_{1,2} \cdot r_1(x) - a_{0,2} \cdot r_0(x) \\
r_3(x) &= \alpha_3 \cdot x r_2(x) - a_{2,3} \cdot r_2(x) - a_{1,3} \cdot r_1(x) - a_{0,3} \cdot r_0(x) \\
&\vdots
\end{aligned}
$$

$$
\begin{bmatrix}
\frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\
\frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\
0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\
0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n}
\end{bmatrix}
$$

Confederate Matrices

$$
\begin{aligned}
r_1(x) &= \alpha_1 \cdot x r_0(x) - a_{0,1} \cdot r_0(x) \\
r_2(x) &= \alpha_2 \cdot x r_1(x) - a_{1,2} \cdot r_1(x) - a_{0,2} \cdot r_0(x) \\
r_3(x) &= \alpha_3 \cdot x r_2(x) - a_{2,3} \cdot r_2(x) - a_{1,3} \cdot r_1(x) - a_{0,3} \cdot r_0(x) \\
&\vdots
\end{aligned}
$$

$$
\begin{bmatrix}
\frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\
\frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\
0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\
0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n}
\end{bmatrix}
$$

Confederate Matrices

$$
\begin{aligned}
r_1(x) &= \alpha_1 \cdot x r_0(x) - a_{0,1} \cdot r_0(x) \\
r_2(x) &= \alpha_2 \cdot x r_1(x) - a_{1,2} \cdot r_1(x) - a_{0,2} \cdot r_0(x) \\
r_3(x) &= \alpha_3 \cdot x r_2(x) - a_{2,3} \cdot r_2(x) - a_{1,3} \cdot r_1(x) - a_{0,3} \cdot r_0(x) \\
&\vdots
\end{aligned}
$$

$$
\begin{bmatrix}
\frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_0}{b_n} \\
\frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_1}{b_n} \\
0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_2}{b_n} \\
0 & 0 & \frac{1}{\alpha_3} & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} - \frac{1}{\alpha_n} \cdot \frac{b_{n-1}}{b_n}
\end{bmatrix}
$$

Motivation for Horner-like Polynomials

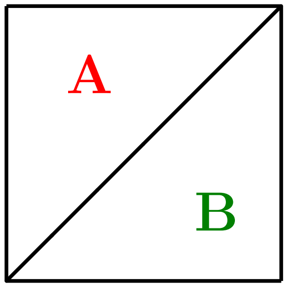
- The confederate matrix $C(b)$ for a polynomial $b(x) = b_0 + b_1 x + \cdots + b_n x_n$ in the **monomial basis** reduces to the companion matrix, and the confederate matrix $C_R(\hat{p}_n)$ for the **Horner polynomials** is:

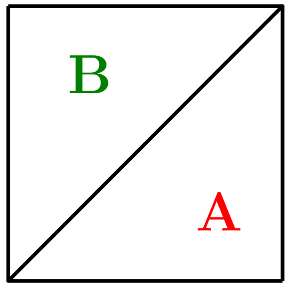
$$\mathbf{C(b)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -b_{n-1} \end{bmatrix} \quad \mathbf{C_R(\hat{p}_n)} = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -b_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

- **Observation**: $\mathbf{C_R(\hat{p}_n)} = \tilde{I} \cdot \mathbf{C(b)}^T \cdot \tilde{I}$

Horner-like Polynomials

➠ A relation between the matrix corresponding to a system of polynomials R and the matrix corresponding to the **Horner-like** polynomials $\hat{R}$.

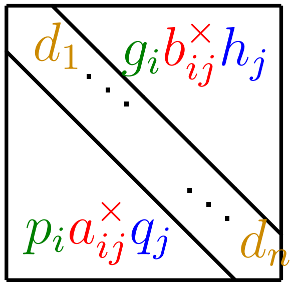




C	$\Rightarrow$ **pertransposition**	$\tilde{I} \cdot C^T \cdot \tilde{I}$
$\{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$		$\{\hat{r}_0(x), \hat{r}_1(x), \ldots, \hat{r}_{n-1}(x)\}$
Polynomial System		Horner-like polynomials

Horner-like Polynomials

➠ **Fact.** A pertransposed quasiseparable matrix is again a quasiseparable matrix.

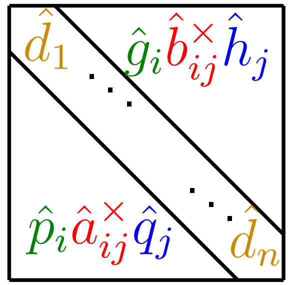


Q/S-Hessenberg Matrix

$\{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$

Polynomial System

$\Rightarrow$
pertransposition



Q/S-Hessenberg Matrix

$\{\hat{r}_0(x), \hat{r}_1(x), \ldots, \hat{r}_{n-1}(x)\}$

Horner-like polynomials

Generators of a Quasiseparable matrix

- Can be represented in terms of their **generators**:

Diagonal entries

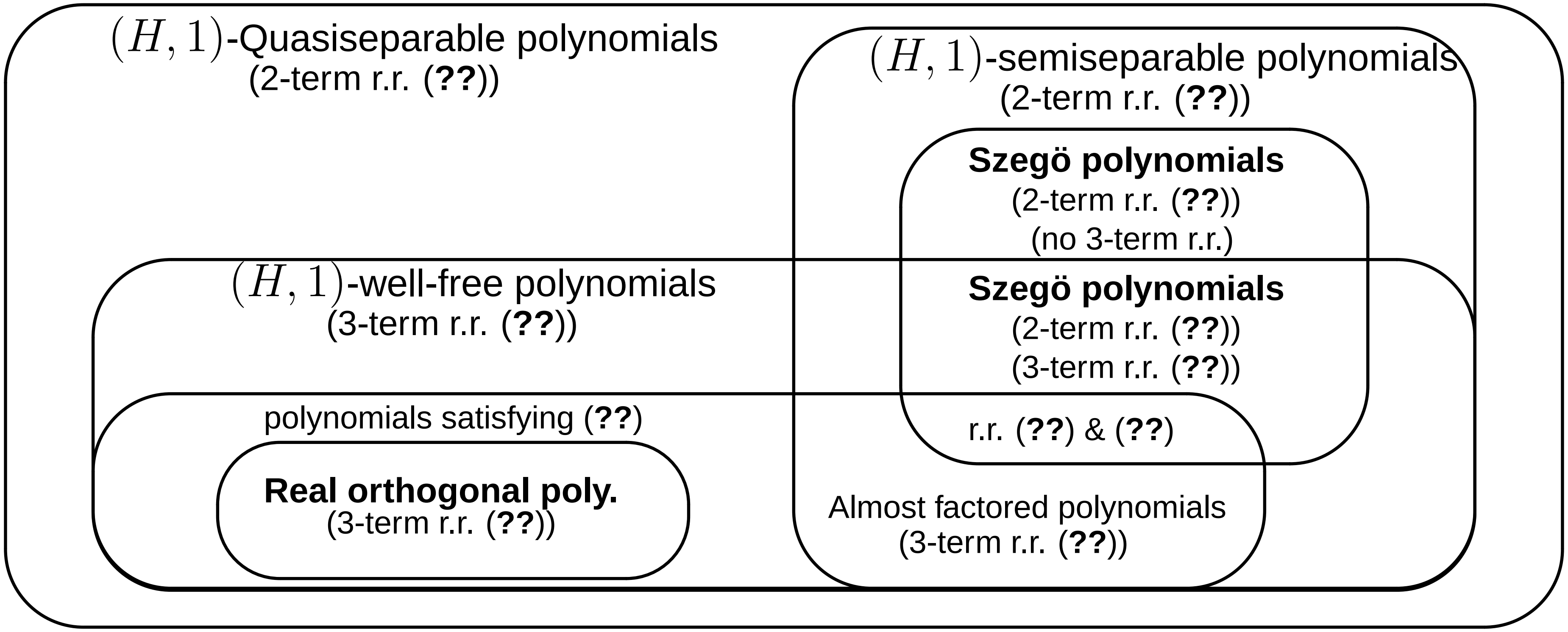
$$d_k \quad k = 1, \ldots, n$$

Lower Generators		**Upper Generators**	
p_k	$k = 2, \ldots, n$	g_k	$k = 1, \ldots, n-1$
a_k	$k = 2, \ldots, n-1$	b_k	$k = 2, \ldots, n-1$
q_k	$k = 1, \ldots, n-1$	h_k	$k = 2, \ldots, n$

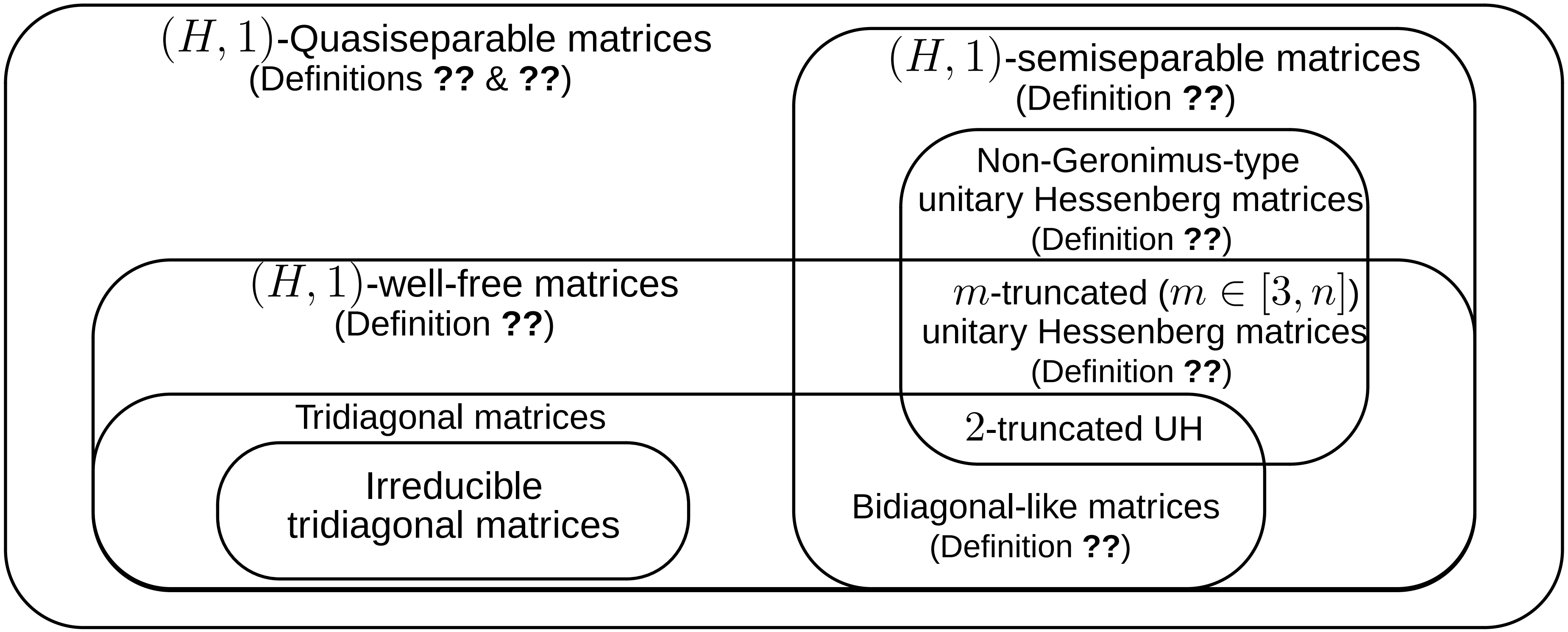
- **Example.** In terms of generators, with $n = 5$,

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

Subclasses of $(H,1)$–quasiseparable polynomials



Subclasses of $(H, 1)$–quasiseparable matrices



Quasiseparable matrices and polynomials

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